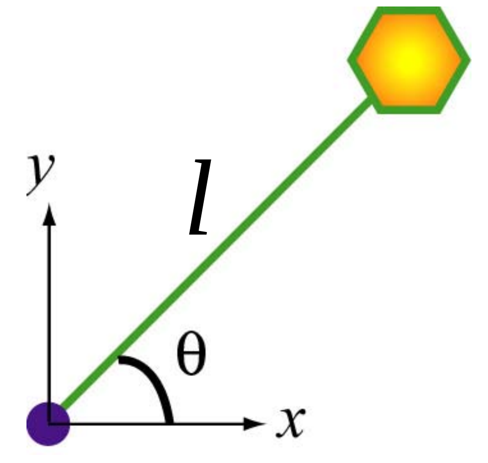
Derivation of dynamics for a simple system: torques and inertia

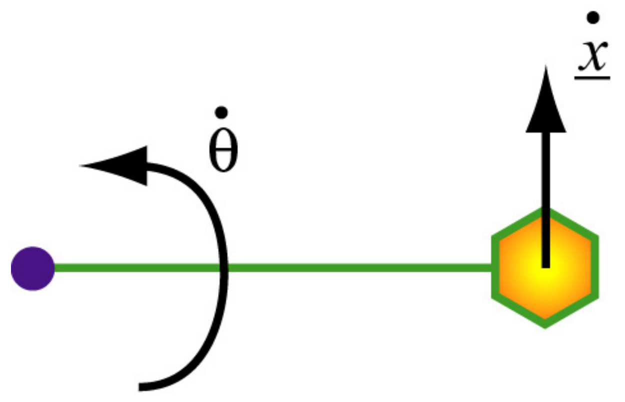
Objective: to describe the forces that act on a rotation mass.
The mass is raised with respect to the ground.



$$\left.\begin{aligned} KE &= \frac{1}{2} m \underline{\dot{x}}^T \underline{\dot{x}} \\ PE &= mgy \end{aligned}\right\} \quad L = KE - PE \qquad \tau = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta}$$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l\cos\theta \\ l\sin\theta \end{bmatrix} \qquad \underline{\dot{x}} = \frac{d\underline{x}}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dx}{d\theta}\frac{d\theta}{dt} \\ \frac{dy}{d\theta}\frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} -l\dot{\theta}\sin\theta \\ l\dot{\theta}\cos\theta \end{bmatrix}$$

linear velocity due to angular velocity



$$\underline{\dot{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l\dot{\theta}\sin\theta \\ l\dot{\theta}\cos\theta \end{bmatrix}$$

$$KE = \frac{1}{2} m \underline{\dot{x}}^T \underline{\dot{x}} = \frac{1}{2} m \left(l^2 \dot{\theta}^2 \sin^2\theta + l^2 \dot{\theta}^2 \cos^2\theta \right) = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$KE = \frac{1}{2} I \dot{\theta}^2$$

inertia: $I = ml^2$

$$\left. \begin{aligned} KE &= \frac{1}{2} m l^2 \dot{\theta}^2 \\ PE &= mgl\sin\theta \end{aligned} \right\} \quad L = KE - PE$$

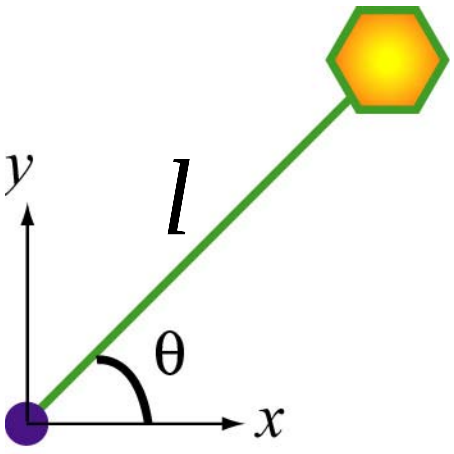
$$\tau = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}} \right) - \frac{dL}{d\theta}$$

$$\tau = ml^2 \ddot{\theta} - mgl\cos\theta$$

Derivation of dynamics for another simple system: forces

$$\left.\begin{aligned} KE &= \frac{1}{2} m \underline{\dot{x}}^T \underline{\dot{x}} \\ PE &= mgy \end{aligned}\right\} \quad L = KE - PE = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$\underline{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \frac{d}{dt}\left(\frac{dL}{d\underline{\dot{x}}}\right) + \frac{dL}{d\underline{x}} = \frac{d}{dt}\begin{bmatrix} \frac{dL}{d\dot{x}} \\ \frac{dL}{d\dot{y}} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} m\dot{x} \\ m\dot{y} \end{bmatrix} = m\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}$$



$$\underline{\dot{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l\dot{\theta}\sin\theta \\ l\dot{\theta}\cos\theta \end{bmatrix} \qquad \underline{\ddot{x}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dfrac{d\dot{x}}{dt} \\ \dfrac{d\dot{y}}{dt} \end{bmatrix}$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d}{dt}\left(-l\dot{\theta}\sin\theta\right) = \frac{d}{dt}\left(-l\dot{\theta}\right)\sin\theta - l\dot{\theta}\frac{d}{dt}\left(\sin\theta\right)$$

$$= -l\ddot{\theta}\sin\theta - l\dot{\theta}\frac{d}{dt}\left(\sin\theta\right) = -l\ddot{\theta}\sin\theta - l\dot{\theta}\frac{d}{d\theta}\left(\sin\theta\right)\frac{d\theta}{dt}$$

$$= -l\ddot{\theta}\sin\theta - l\dot{\theta}\dot{\theta}\cos\theta = -l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta$$

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d}{dt}\left(\dot{\theta}\cos\theta\right) = l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -l\sin\theta \\ l\cos\theta \end{bmatrix}\ddot{\theta} + \begin{bmatrix} -l\cos\theta \\ -l\sin\theta \end{bmatrix}\dot{\theta}^2$$

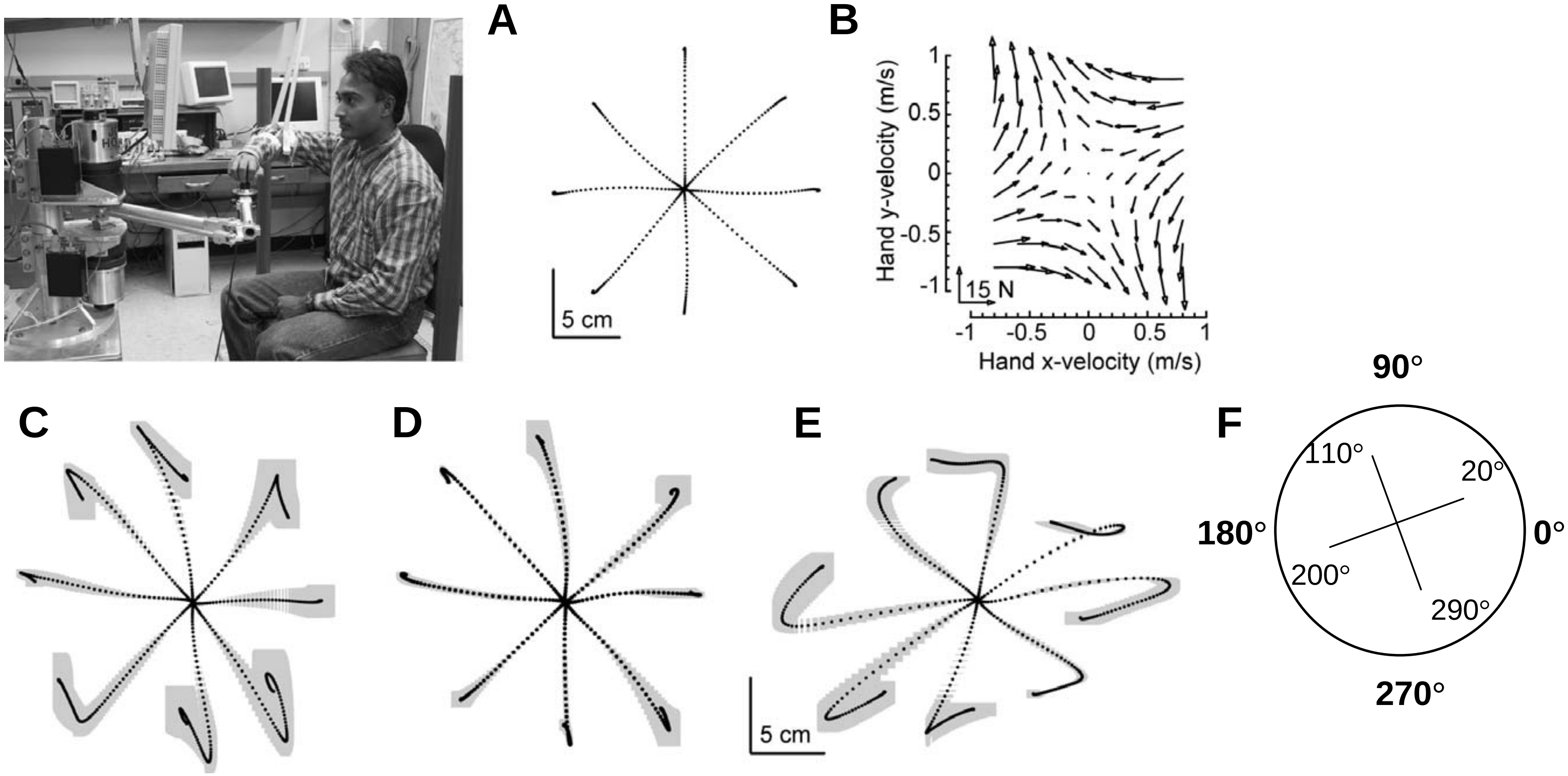
$$\underline{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = ml \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \ddot{\theta} + \underbrace{ml \begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix} \dot{\theta}^2}_{\text{centripetal force}} - \begin{bmatrix} 0 \\ mgy \end{bmatrix}$$

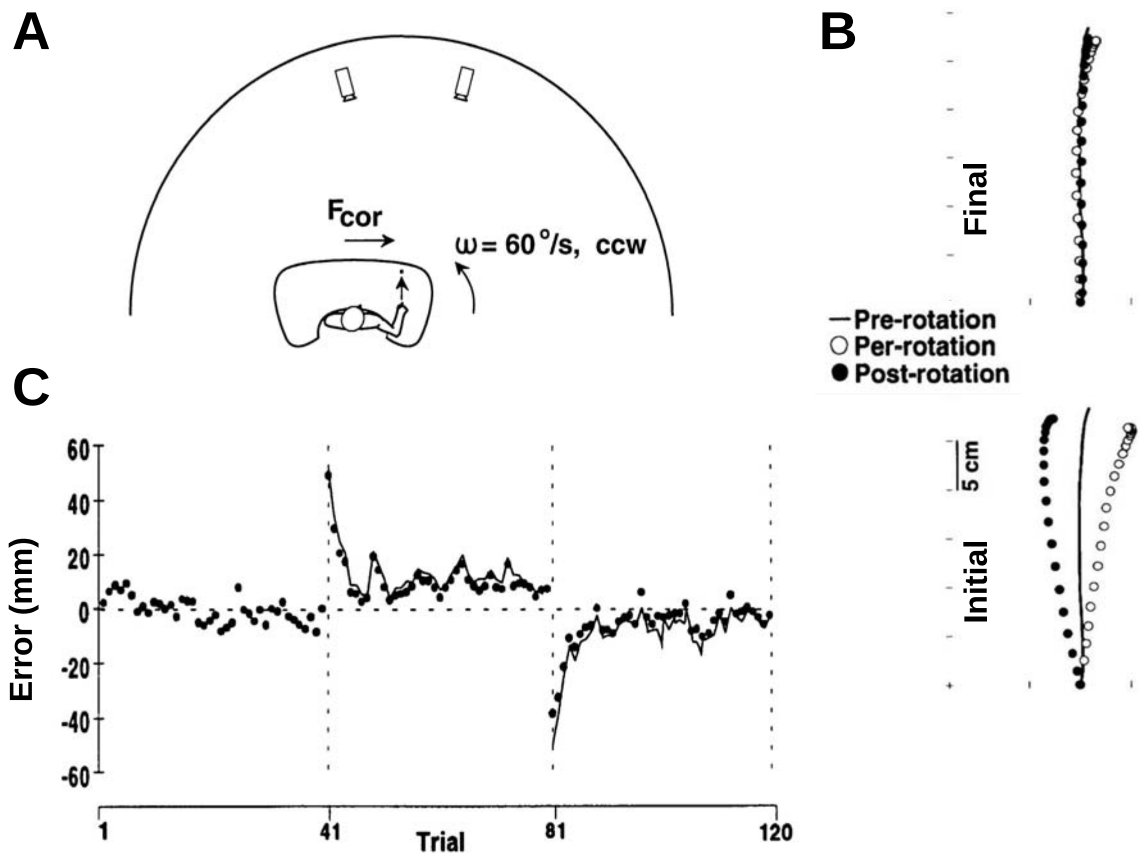
$$\underline{f} - \underbrace{ml \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \ddot{\theta} - ml \begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix} \dot{\theta}^2}_{\text{force that the mass of the cup “produces” to resist motion}} = 0$$

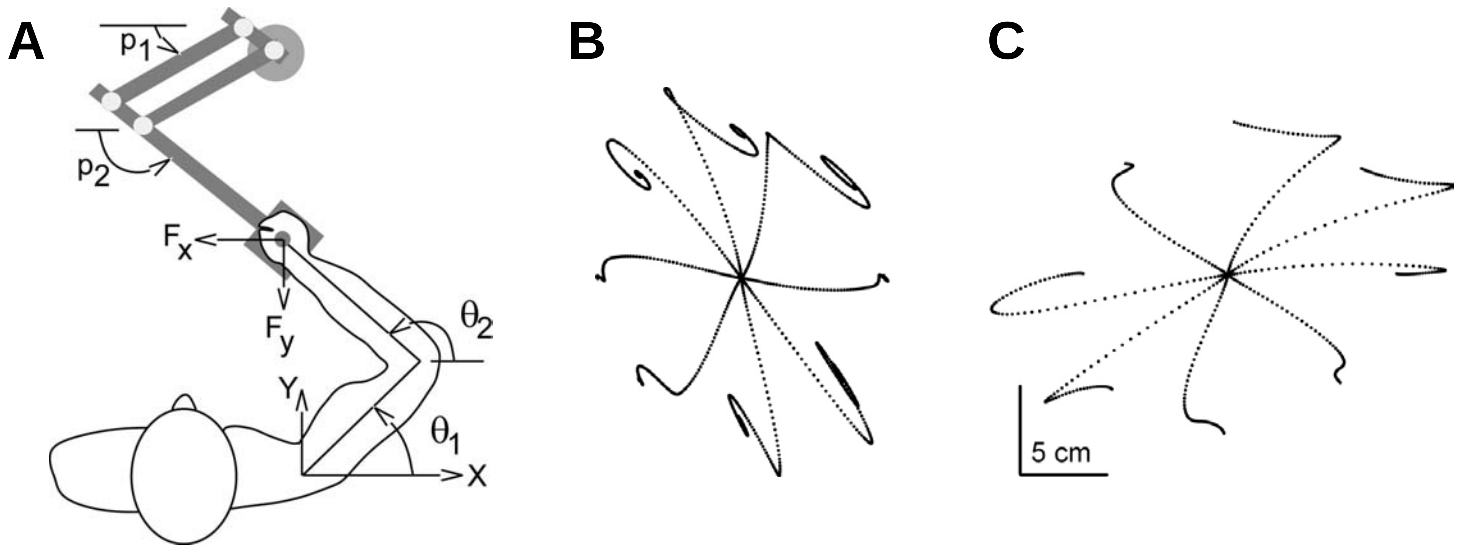
$\underline{f}$: force that my torque motor must make to produce motion

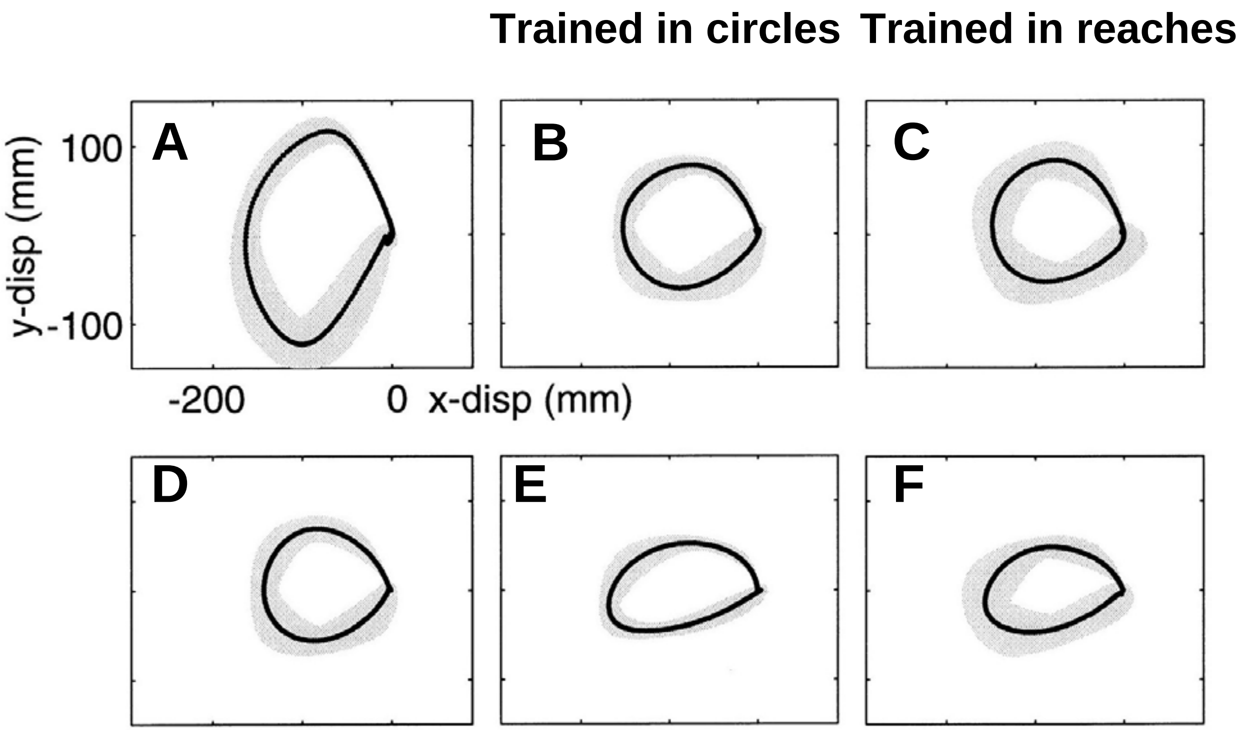
The child in the cup feels the force “produced” by his mass: she feels a centripetal force pushing her outward as she rotates.

Gravity also produces a force on our body. Simulators use gravity to fool the brain into thinking it is feeling a force due to motion.

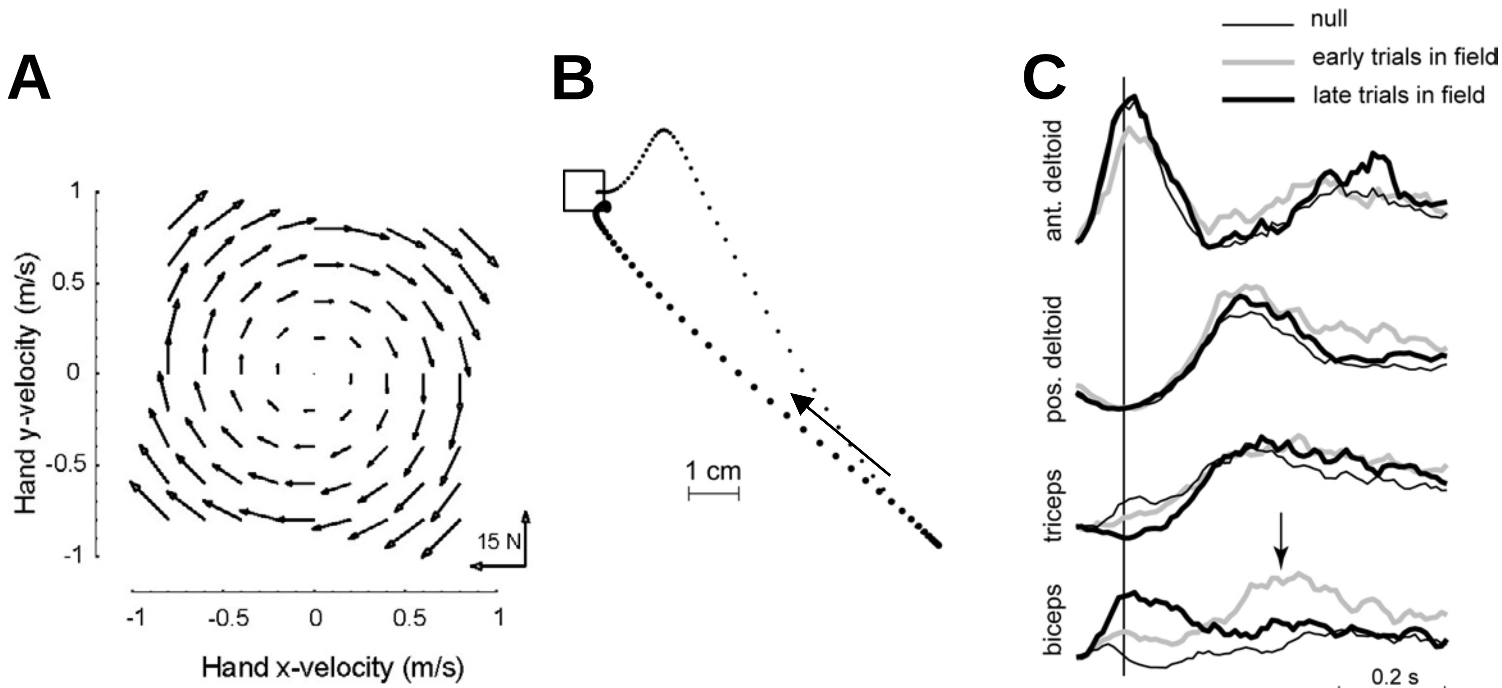

A
B
5 cm
Hand y-velocity (m/s)
Hand x-velocity (m/s)
15 N
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
C
D
E
F
5 cm
90°
180°
0°
270°
110°
20°
200°
290°


A
Fcor
ω = 60 °/s, ccw
B
Final
Initial
5 cm
Pre-rotation
Per-rotation
Post-rotation
C
Error (mm)
60
40
20
0
-20
-40
-60
Trial
1
41
81
120

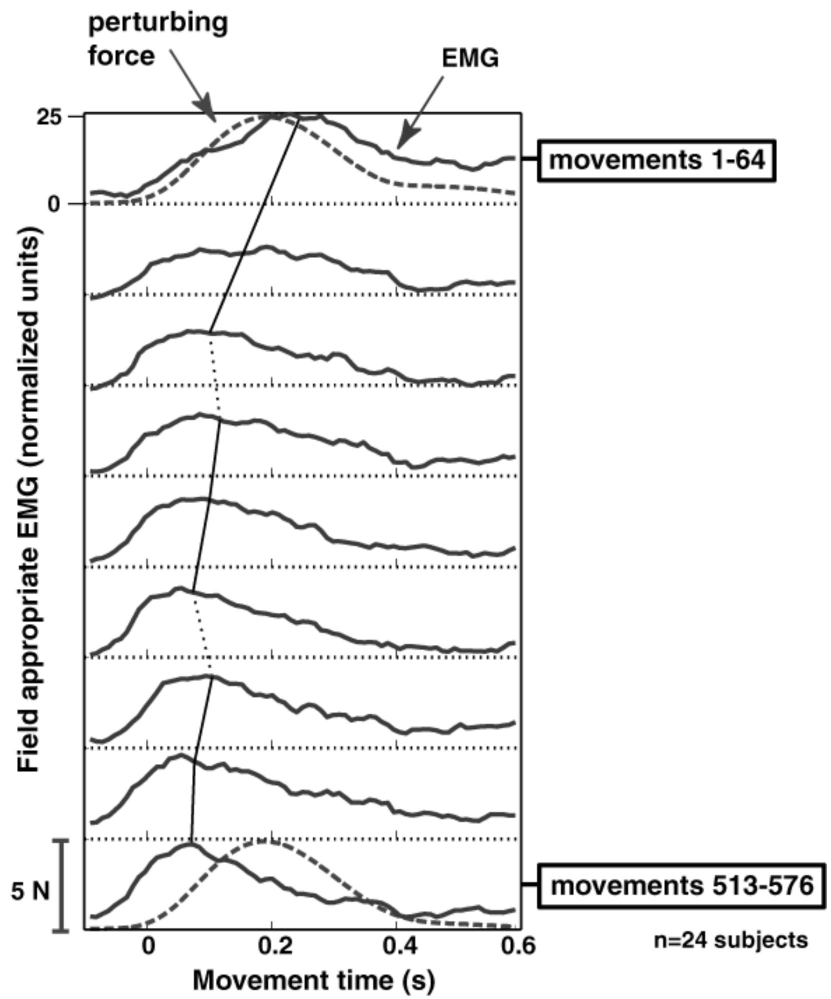

A
p1
p2
Fx
Fy
Y
X
θ2
θ1
B
C
5 cm



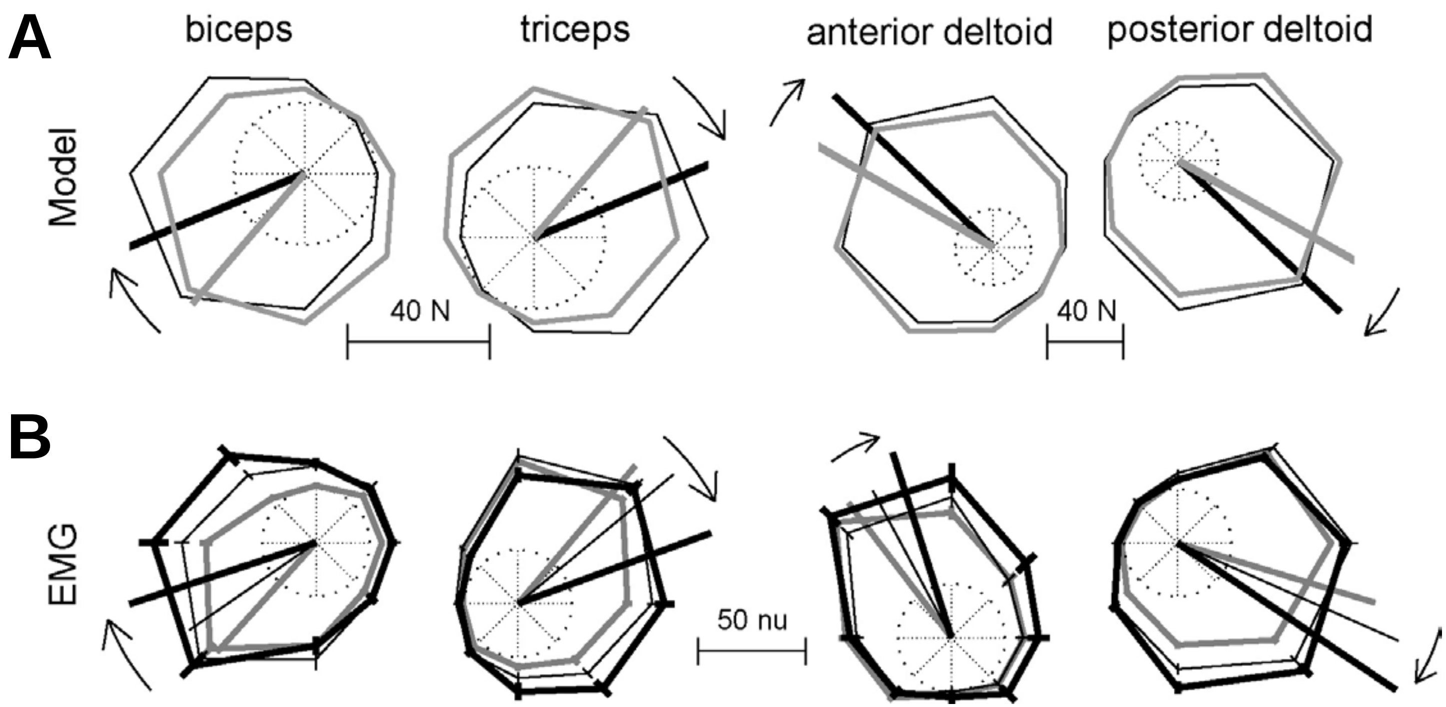
Trained in circles
Trained in reaches
A
B
C
D
E
F
100
-100
y-disp (mm)
-200
0
x-disp (mm)

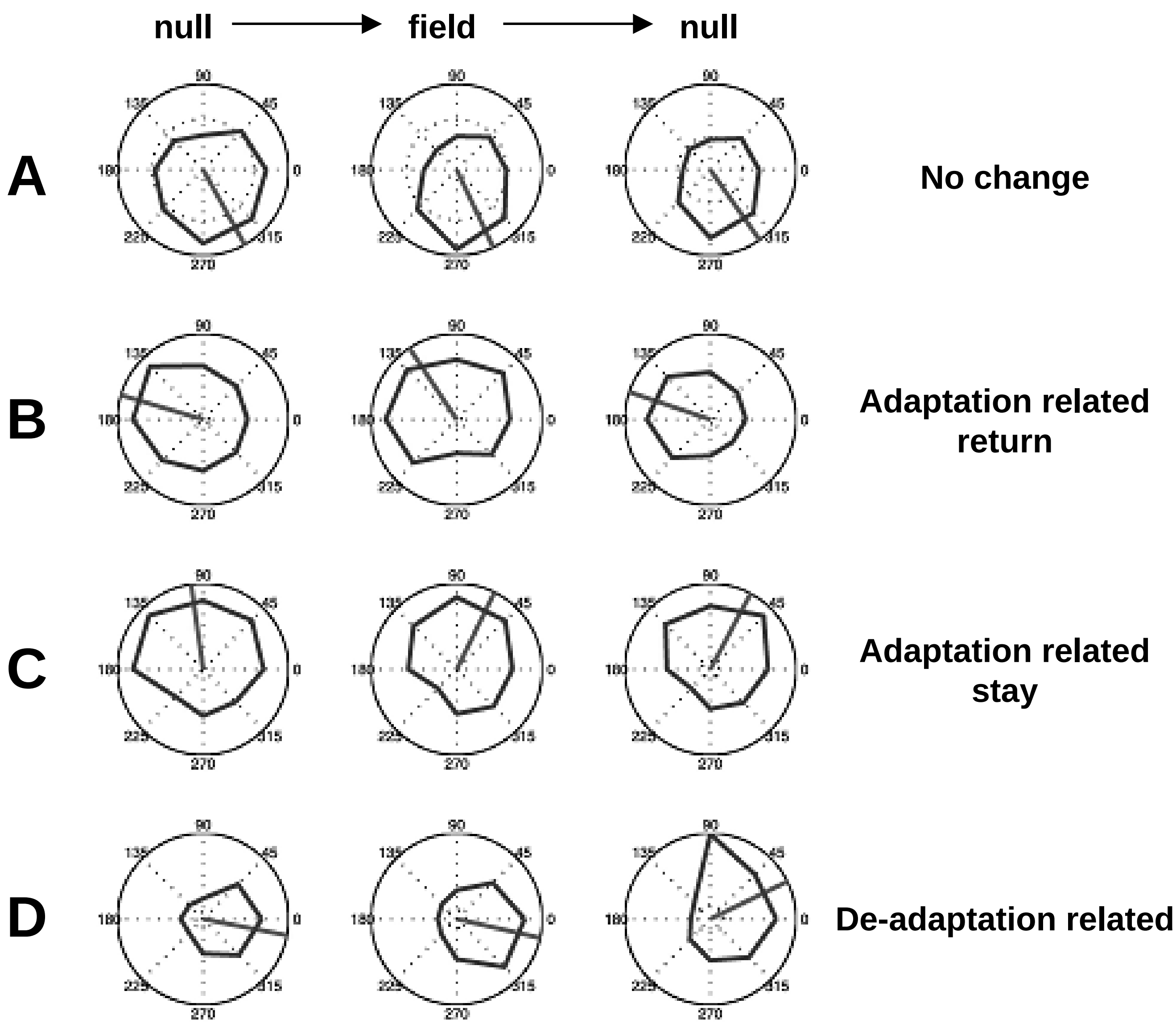


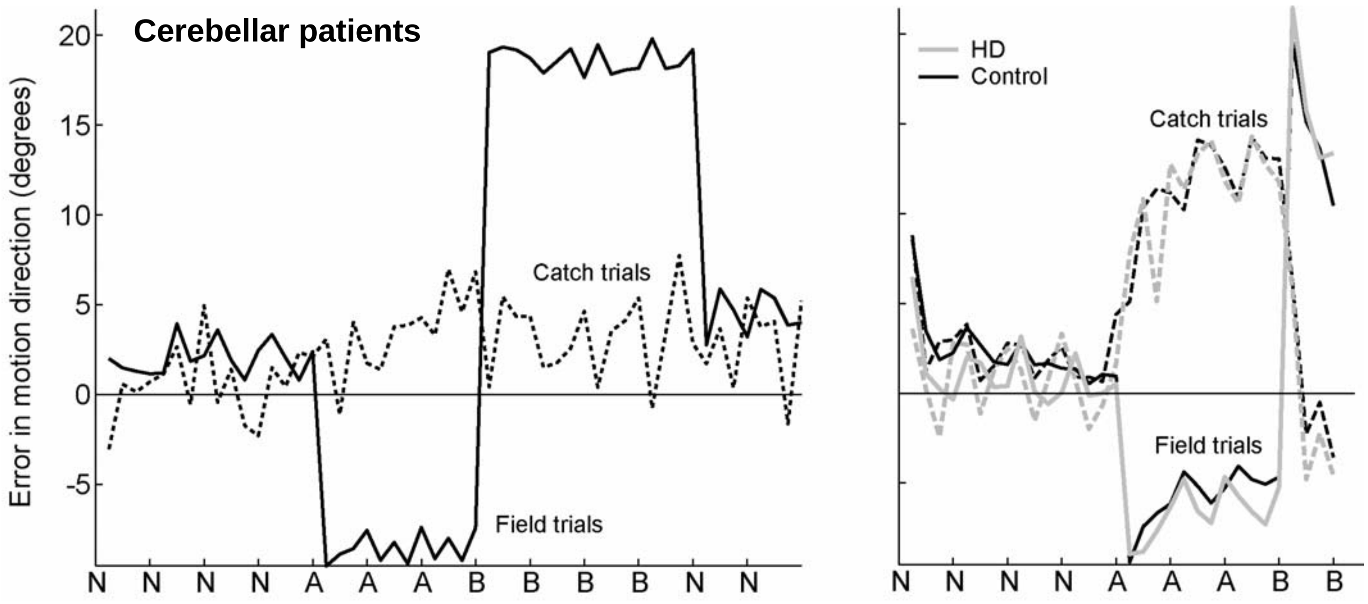
A
B
C
Hand y-velocity (m/s)
Hand x-velocity (m/s)
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
15 N
1 cm
null
early trials in field
late trials in field
ant. deltoid
pos. deltoid
triceps
biceps
0.2 s



perturbing
force
EMG
25
0
Field appropriate EMG (normalized units)
movements 1-64
movements 513-576
5 N
0
0.2
0.4
0.6
Movement time (s)
n=24 subjects


A
B
biceps
triceps
anterior deltoid
posterior deltoid
Model
EMG
40 N
40 N
50 nu


null
field
null
A
B
C
D
No change
Adaptation related return
Adaptation related stay
De-adaptation related



Cerebellar patients
Error in motion direction (degrees)
20
15
10
5
0
-5
Catch trials
Field trials
N N N N A A A B B B B N N
HD
Control
Catch trials
Field trials
N N N N A A A B B