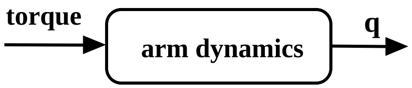
What is dynamics?

A description of how forces acting on a system result in motion of that system.



Example: A ball of mass m is held 20 m off the ground. The force acting on the ball is the force of gravity: f=-mg where g=9.8 m/s^2. If we drop the ball, its dynamics are describe by:

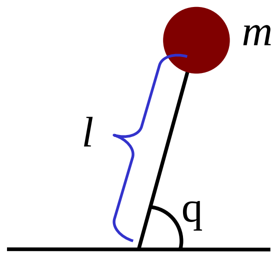
$$x(0) = 20\text{ m}$$

$$\dot{x}(0) = 0\text{ m/s}$$

$$f = -mg$$

$$\ddot{x}(t) = \frac{f}{m} = -g = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2$$

Example: Dynamics of a single joint system with mass m and length l.



$$\ddot{q} = \frac{1}{ml^2}(\tau - mgl\cos q)$$

Path of motion of a system is one that minimizes an energy cost

Imagine a point mass that is at position x_1 at time t_1 and ends up at position x_2 at t_2, for example: a ball falling from a height. The trajectory that it follows to get to x_2 is only one of an infinite number of pathways that it could have followed. But the point mass will always follow that same trajectory $x(t)$, given the same initial conditions. What is so special about the trajectory $x(t)$ that it actually does follow?

The trajectory $x(t)$ minimizes the following cost function:

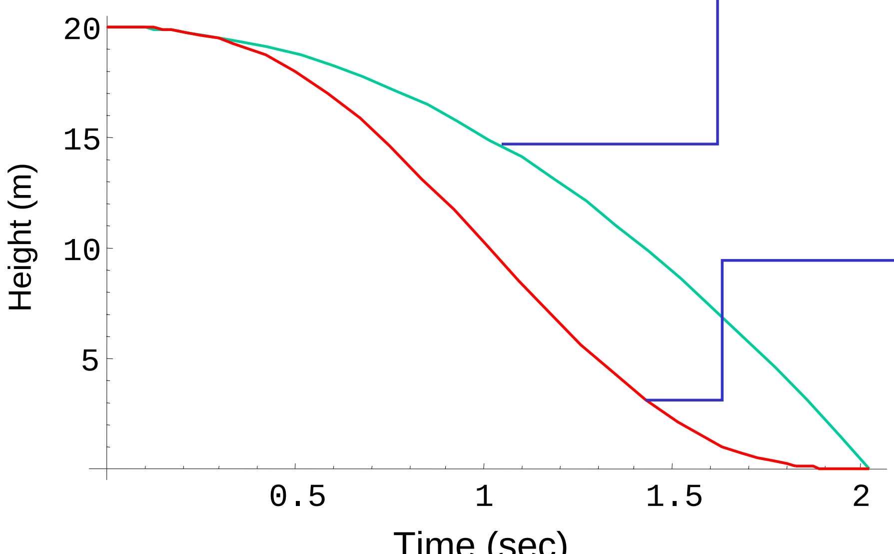
$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

Kinetic energy

Potential energy

Lagrangian

For a point mass: $\begin{cases} KE = \frac{1}{2} m\dot{x}^2 \\ PE = mgx \end{cases}$



$$\ddot{x}(t) = -9.8 \quad \rightarrow \quad x(t) = 20 - 4.9t^2$$

$$H(x(t)) = \int_0^{2.02} \left(\frac{1}{2} m\dot{x}(t)^2 - mgx(t)\right) dt = -264$$

$$x(t) = 20 + 20(-1.21t^3 + 0.9t^4 - 0.178t^5)$$

$$H(x(t)) = \int_0^{2.02} \left(\frac{1}{2} m\dot{x}(t)^2 - mgx(t)\right) dt = -113$$

Solving the functional for a point mass

$$H(x(t)) = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} \frac{1}{2} m\dot{x}^2 - mgx\, dt$$

$$x(t) \to x(t) + e\eta(t)$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2} m(\dot{x} + e\dot{\eta})^2\, dt - \int_{t_1}^{t_2} mg(x + e\eta)\, dt$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2} m(\dot{x} + e\dot{\eta})^2\, dt - \int_{t_1}^{t_2} mg(x + e\eta)\, dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = \int_{t_1}^{t_2} m(\dot{x} + e\dot{\eta})\dot{\eta}\, dt - \int_{t_1}^{t_2} mg\eta\, dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = \int_{t_1}^{t_2} m\dot{x}\dot{\eta}\, dt - \int_{t_1}^{t_2} mg\eta\, dt$$

$$u = \dot{x} \quad dv = \dot{\eta}\, dt \quad du = \ddot{x}dt \quad v = \eta$$

$$\int_{t_1}^{t_2} m\dot{x}\dot{\eta}\, dt = m\dot{x}\eta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\eta\ddot{x}dt = -\int_{t_1}^{t_2} m\ddot{x}\eta dt$$

$$\left.\frac{dH}{de}\right|_{e=0} = -\int_{t_1}^{t_2} m\ddot{x}\eta dt - \int_{t_1}^{t_2} mg\eta\, dt = 0$$

$$-m\ddot{x} - mg = 0 \quad \to \quad \boxed{-mg = m\ddot{x}}$$

General solution for the functional

$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} \underbrace{L(x, \dot{x}, t)}_{\text{Lagrangian}}\, dt$$

$$\boxed{\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0}$$

The solution to the calculus of variation approach to minimize $H(x)$

Example: dynamics of a point mass

$$L = \frac{1}{2} m\dot{x}^2 - mgx$$

$$\frac{dL}{d\dot{x}} = m\dot{x} \quad \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = m\ddot{x} \quad \frac{dL}{dx} = -mg$$

$$m\ddot{x} + mg = 0 \;\rightarrow\; -mg = m\ddot{x}$$

If there are external forces (from motors, muscles) acting on the system:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = F$$

The primary problem in dynamics is to find an expression for the kinetic energy of the system.