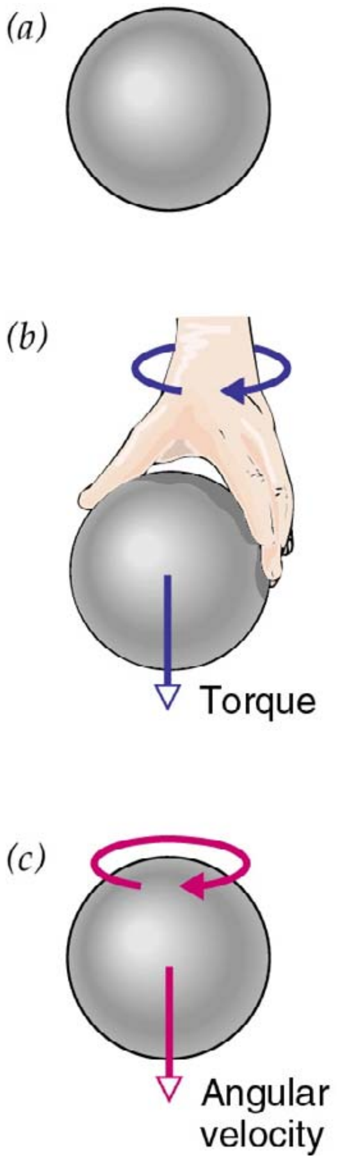
Torque and angular velocity as vectors

"Right-hand rule" describes the direction of the vector



Principle of Virtual Work

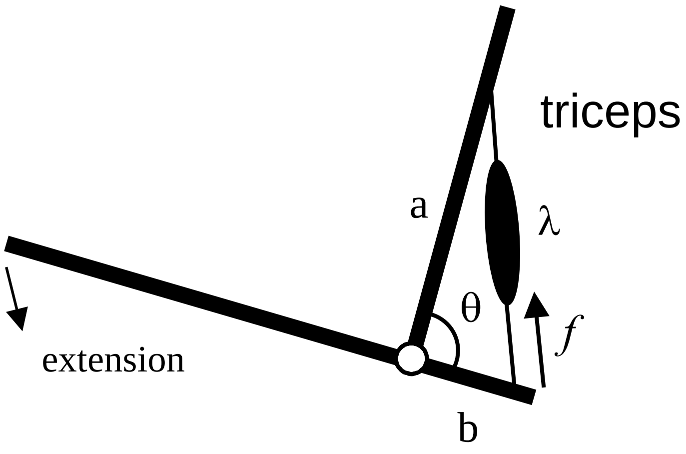
$$\tau\ \Delta\theta\ =\ -f\ \Delta\lambda$$

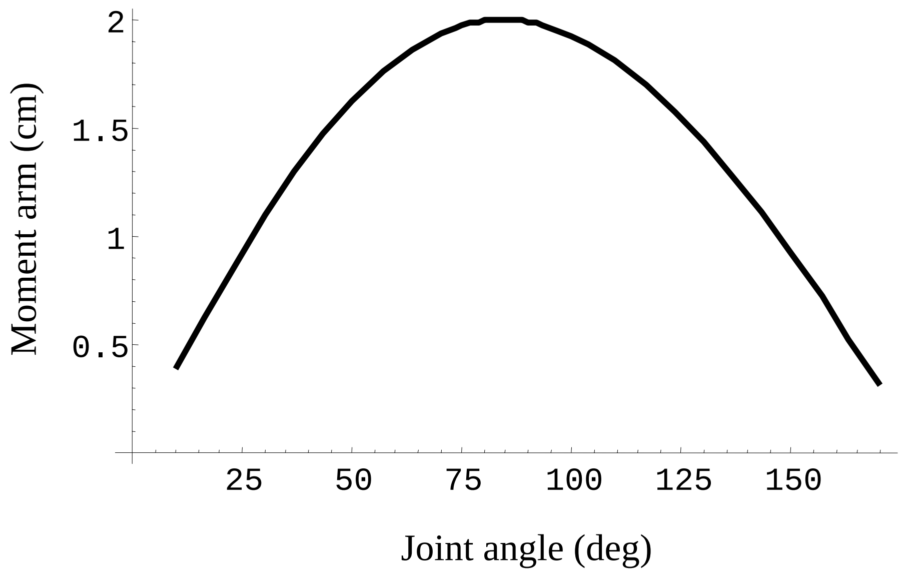
$$\tau = -\frac{\Delta\lambda}{\Delta\theta} f$$

$$\tau = -\frac{d\lambda}{d\theta} f$$

$$\lambda = \sqrt{a^2 + b^2 - 2ab\cos(\theta)}$$

$$\frac{d\lambda}{d\theta} = \frac{ab\sin(\theta)}{\sqrt{a^2 + b^2 - 2ab\cos(\theta)}}$$





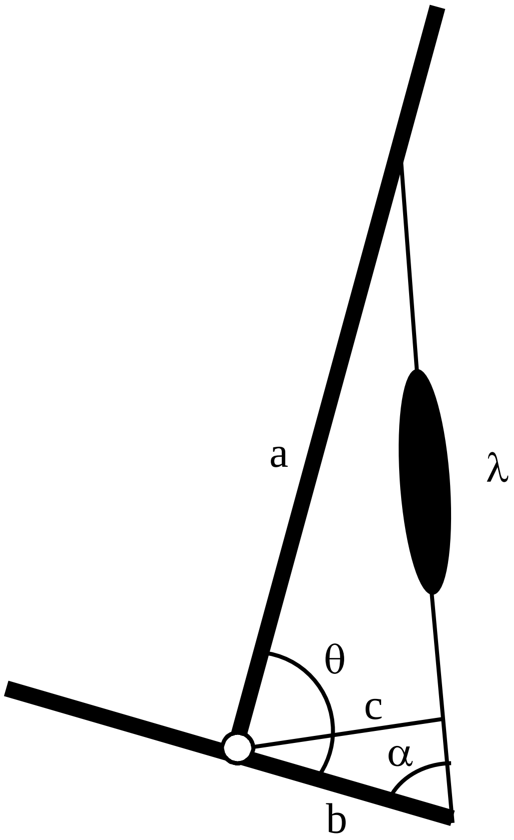
Moment arms

$$c = b\sin(\alpha)$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\theta)}{\lambda}$$

$$c = \frac{ab\sin(\theta)}{\lambda} = \frac{ab\sin(\theta)}{\sqrt{a^2 + b^2 - 2ab\cos(\theta)}}$$

$$\frac{d\lambda}{d\theta} = c$$



$$-f^T d\lambda = \tau^T d\theta$$

$$J_\lambda \equiv \frac{d\lambda}{d\theta} \quad \text{Jacobian}$$

$$-f^T J_\lambda \, d\theta = \tau^T d\theta$$

$$-f^T J_\lambda = \tau^T \quad \rightarrow \quad -(f^T J_\lambda)^T = \tau \quad \rightarrow \quad \boxed{\tau = -J_\lambda{}^T f}$$

$$\lambda = \sqrt{d^2 + c^2 + 2dc\cos(\beta + \theta_1)}$$

$$\beta = \arcsin\left(\frac{b\sin(\theta_2)}{c}\right)$$

$$c = \sqrt{a^2 + b^2 + 2ab\cos(\theta_2)}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

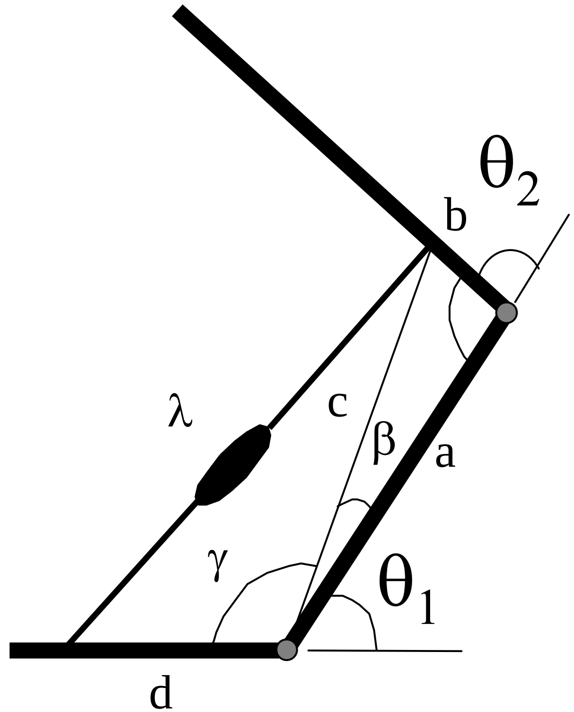
$$\lambda = \sqrt{d^2 + c^2 + 2dc\left[\cos(\beta)\cos(\theta_1) - \sin(\beta)\sin(\theta_1)\right]}$$

$$\cos x = \sqrt{1 - \sin^2 x} \quad \rightarrow \quad \cos(\arcsin a) = \sqrt{1 - a^2}$$

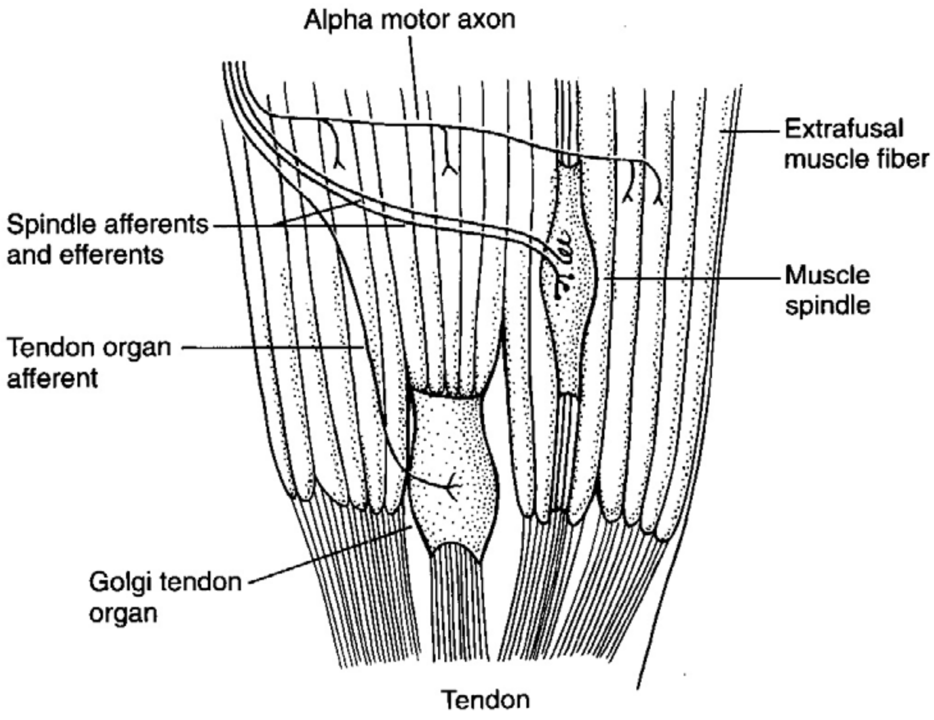
$$\lambda = \sqrt{d^2 + c^2 + 2dc\left(\sqrt{1 - \frac{b^2\sin^2(\theta_2)}{c^2}}\cos(\theta)_1 - \frac{b\sin(\theta_2)}{c}\sin(\theta_1)\right)}$$

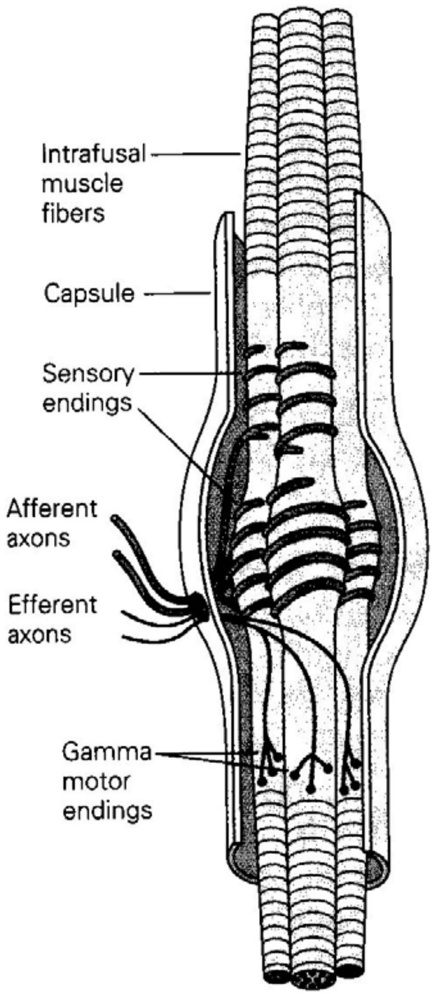
$$J_\lambda = \frac{d\lambda}{d\theta} = \begin{bmatrix} \frac{d\lambda}{d\theta_1} & \frac{d\lambda}{d\theta_2} \end{bmatrix}$$

$$\tau = -J_\lambda^{\,T} f$$

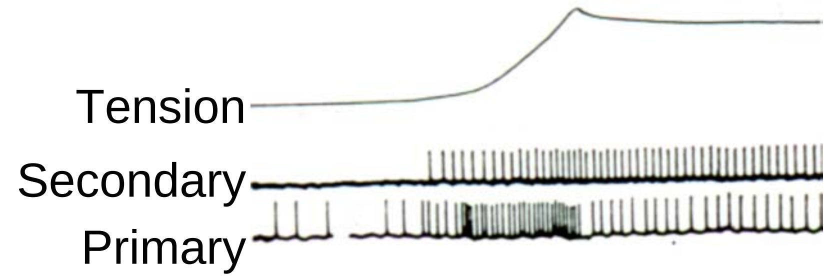


Muscle's Sensory System

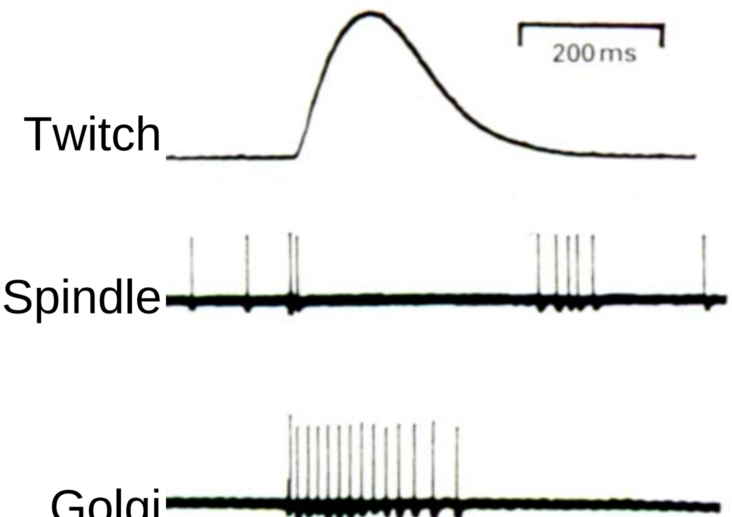




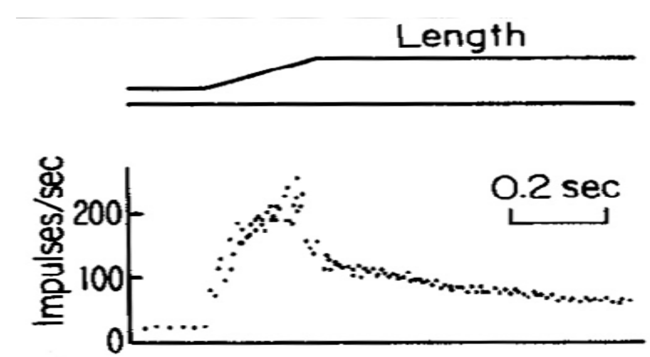
Response of spindle afferents to a lengthening of the muscle



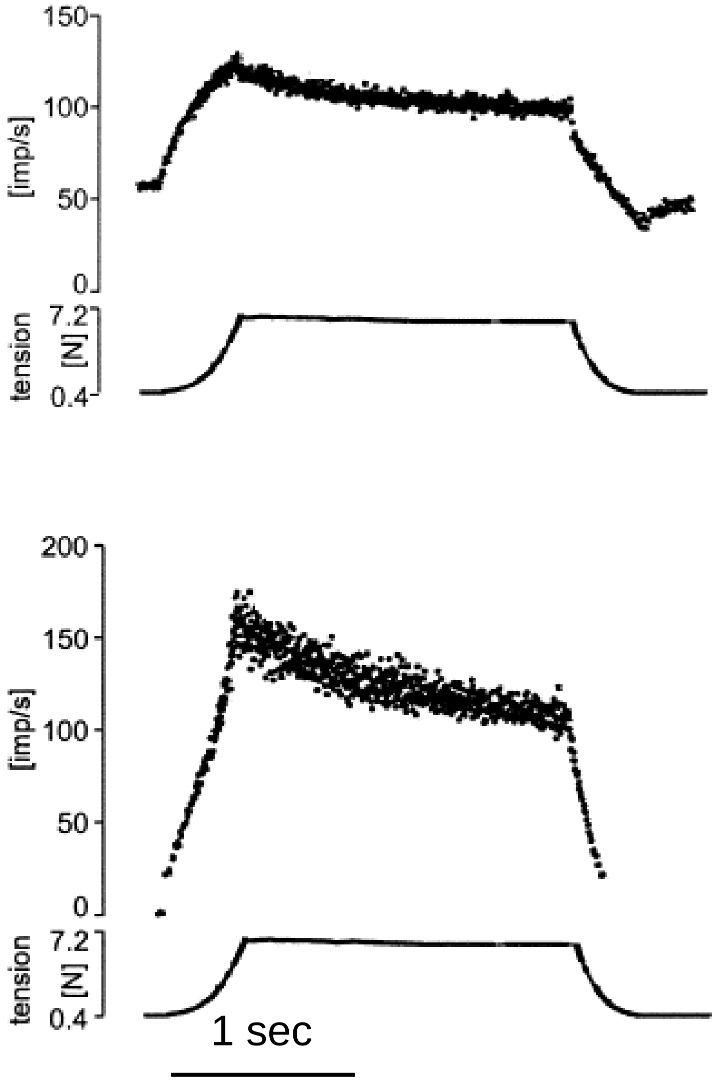
Response of spindle and Golgi afferents to a single action potential delivered to the muscle.



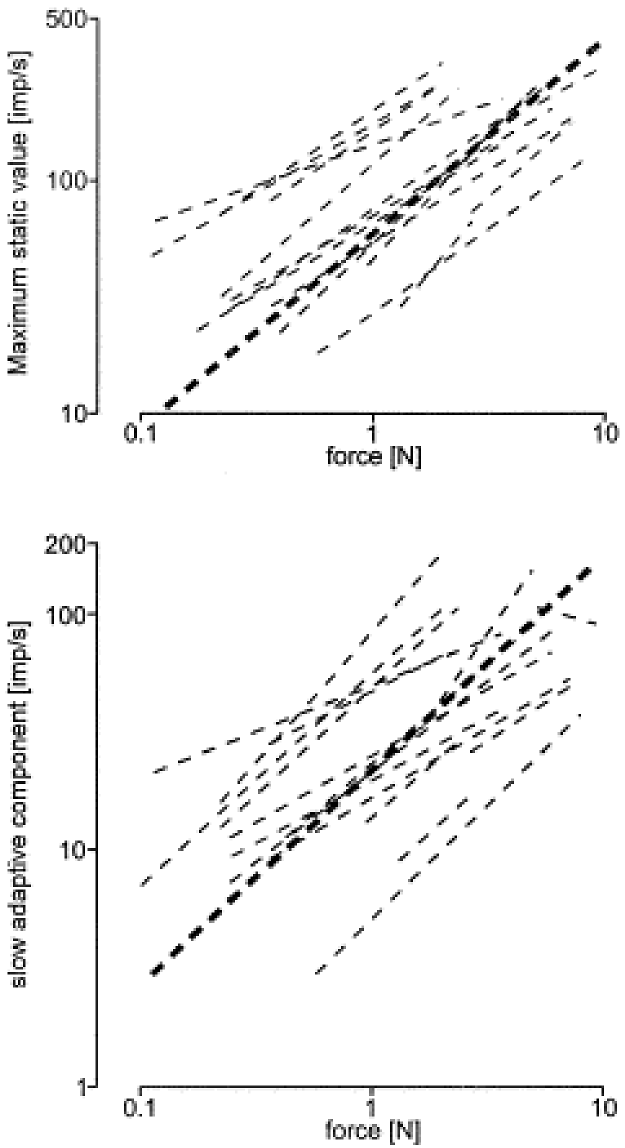
Response of a spindle afferent to lengthening of the muscle



Response of two Golgi tendon organs in a cat muscle



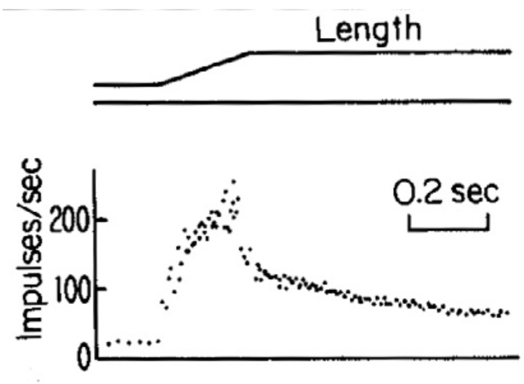
Peak and steady-state discharge of Golgi tendon organs as a function of force in a cat muscle

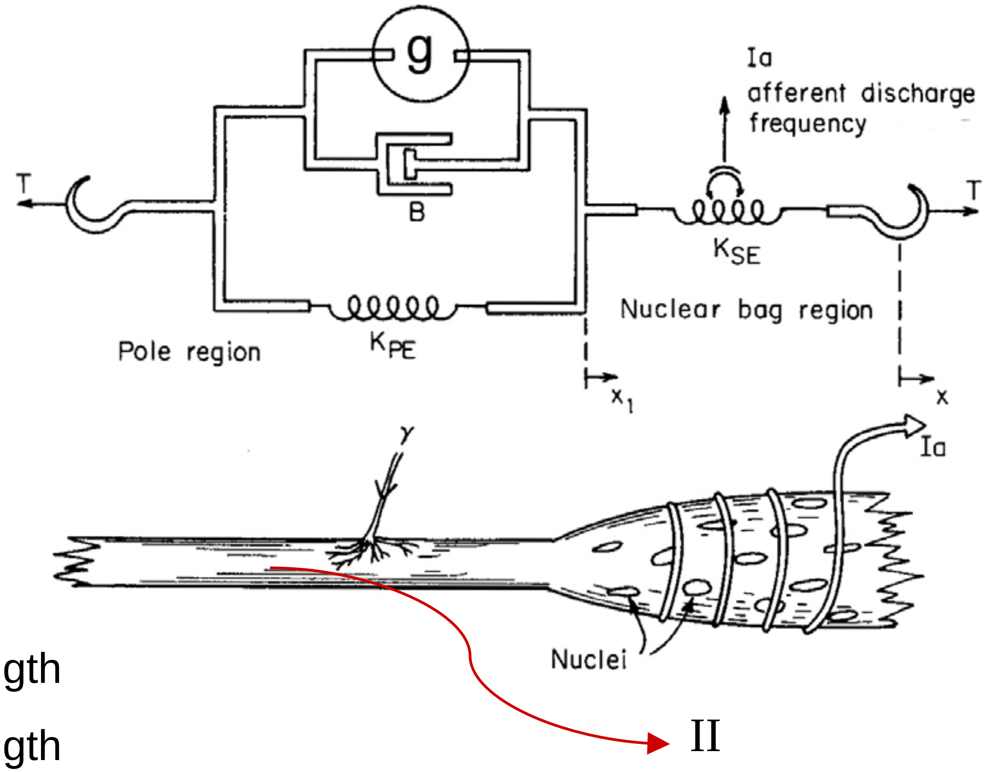


Each dash line is the fit to a single Golgi tendon organ. Heavy dash line is average for all units.

Model of a Muscle Spindle Afferent

Response of a muscle spindle 1a afferent to a stretch





x_1: length of PE element beyond resting length

x_2: length of SE element beyond resting length

x: length of spindle beyond resting length

g(t): input from γ-motor neuron

S_{1a}(t): discharge of group Ia spindle afferent, assumed to be proportional to length of SE element

$$S_{1a}(t) = a(x - x_1)$$

S_2(t): discharge of group II spindle afferent, assumed to be proportional to length of PE element

$$S_2(t) = ax_1$$

$$T = K_{se}(x - x_1)$$

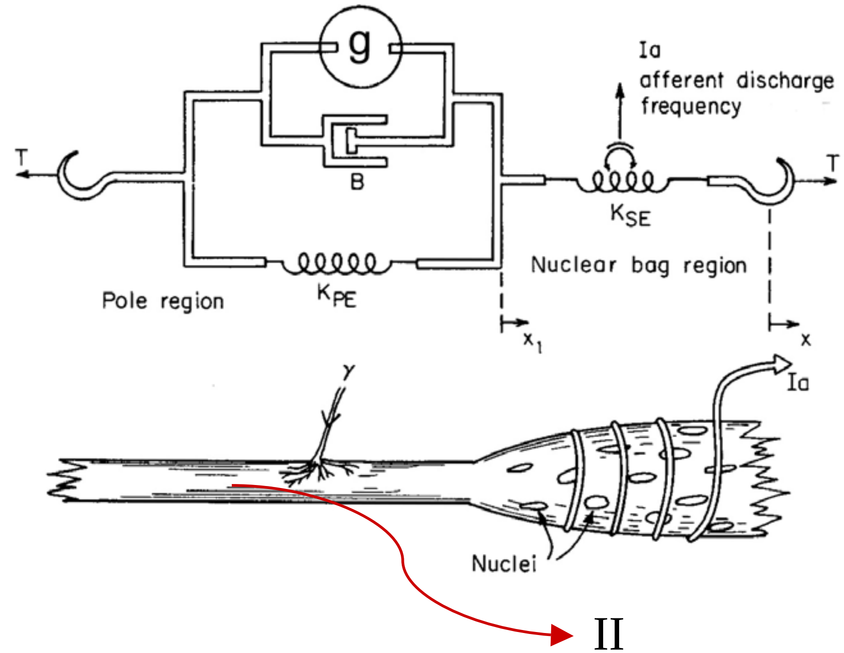
$$x_1 = x - \frac{T}{K_{se}} \quad \rightarrow \quad \dot{x}_1 = \dot{x} - \frac{\dot{T}}{K_{se}}$$

$$T = K_{pe}x_1 + b\dot{x}_1 + g$$

$$T = K_{pe}\left(x - \frac{T}{K_{se}}\right) + b\left(\dot{x} - \frac{\dot{T}}{K_{se}}\right) + g$$

$$\left(1 + \frac{K_{pe}}{K_{se}}\right)T + \frac{b}{K_{se}}\dot{T} = b\dot{x} + K_{pe}x + g$$

$$\dot{T} = \frac{K_{se}}{b}\left(b\dot{x} + K_{pe}x + g\right) - \left(\frac{K_{se} + K_{pe}}{b}\right)T$$



$$\dot{T} = \frac{K_{se}}{b}\left(b\dot{x} + K_{pe}x + g\right) - \left(\frac{K_{se} + K_{pe}}{b}\right)T$$

$$S_{1a}(t) = a(x - x_1) = \frac{aT}{K_{se}}$$

$$S_2(t) = ax_1 = a\left(x - \frac{T}{K_{se}}\right)$$

$$K_{se} = 35 \text{ g/cm}$$

$$K_{pe} = 5 \text{ g/cm}$$

$$b = 10 \text{ g.sec/cm}$$

$$a = 220$$

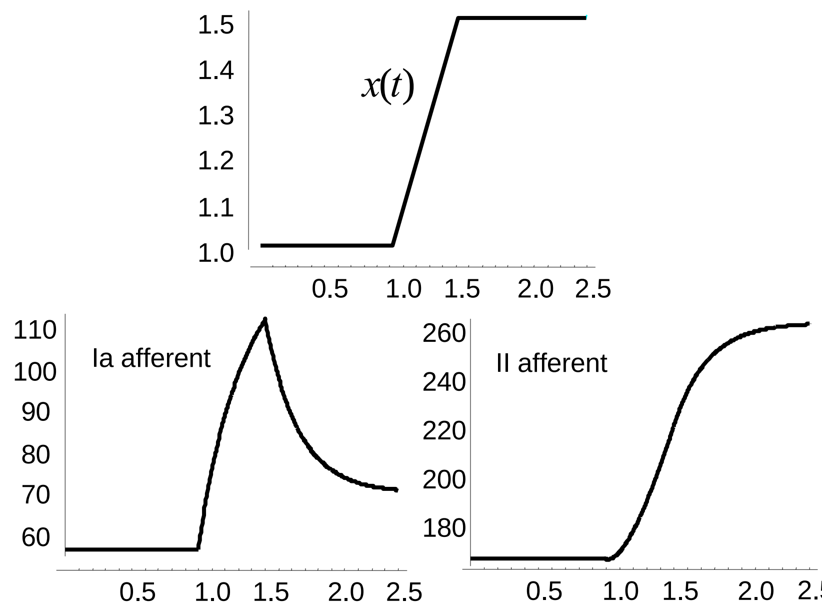
$$g(t) = 0$$

$$x(t = 0) = 0$$

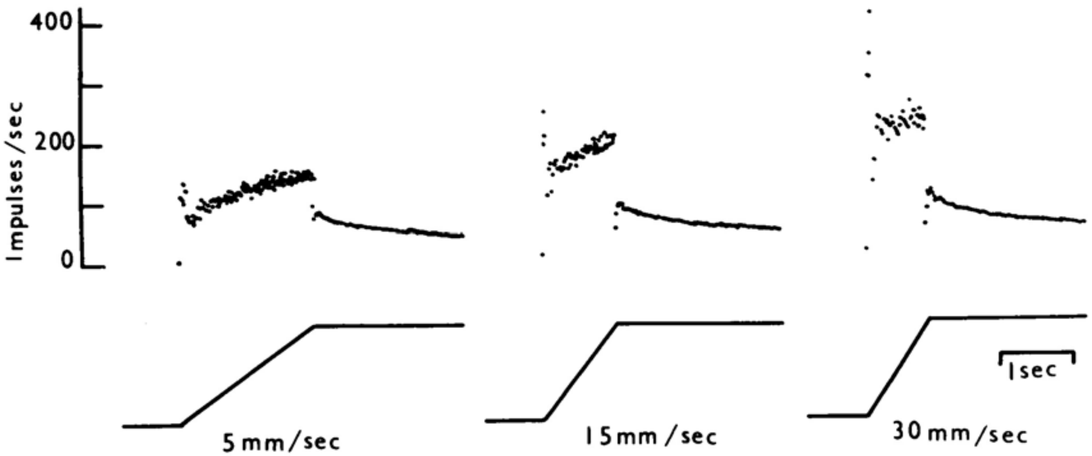
$$\dot{x}(t = 0) = 0$$

$$T(t = 0) = 0$$

The muscle spindle is pulled from rest by a stretch that lasts 0.2 seconds and lengthens the spindle by 1 cm. It is held at this length for 1 second.

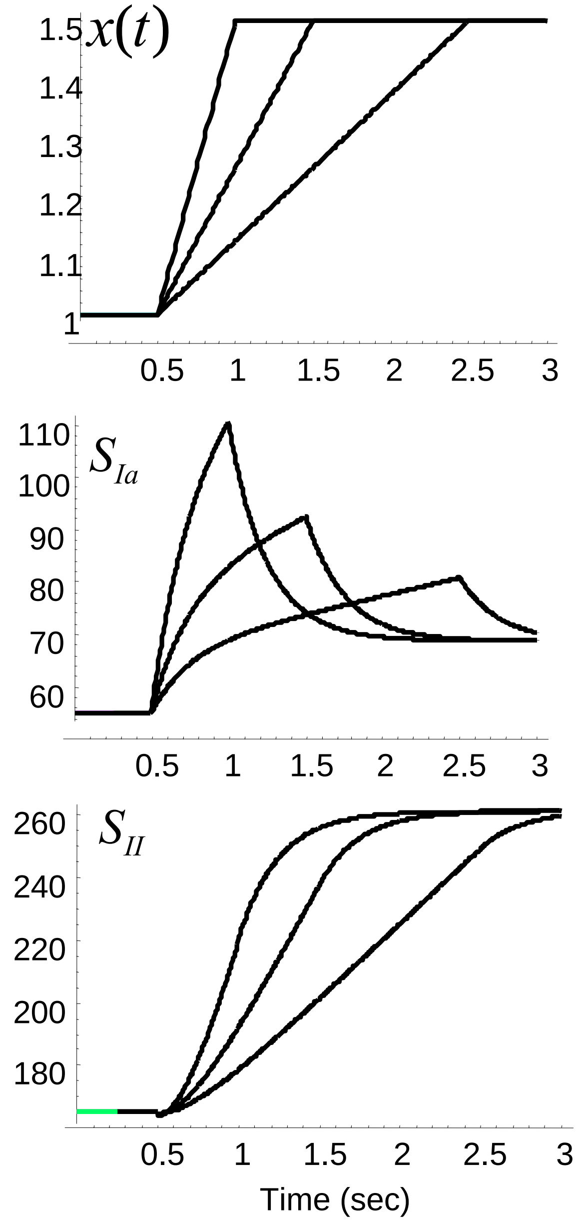


Recordings from primary (Group Ia) muscle-spindle afferents of a cat soleus muscle in response to a 6-mm stretch at various speeds of lengthening



Recordings from 1a spindle afferents of a cat soleus muscle (de-efferented) in response to a 6 mm stretch.

Matthews, *Handbook of Physiol* 1982



Role of the γ -motor neuron input to the spindle

$$\dot{T} = \frac{K_{se}}{b}\left(b\dot{x} + K_{pe}x + g\right) - \left(\frac{K_{se} + K_{pe}}{b}\right)T$$

Initial conditions in the spindle: $\dot{T} = 0; \dot{x} = 0; \quad T(0) = \dfrac{K_{se}(K_{pe}x(0) + g(0))}{K_{se} + K_{pe}}$

We want the extrafusal muscle to change length by amount $x_d(t)$.

We set α-motor neuron input to muscle, and it changes length.

We want to know if the length change was along desired trajectory.

What to do: set $g(t)$ so that as the spindle length changes, if it changes

along the desired trajectory, there is no change in spindle tension, i.e., $\dot{T}(t) = 0$.

$$0 = \frac{K_{se}}{b}\left(b\dot{x}_d + K_{pe}x_d + g\right) - \left(\frac{K_{se} + K_{pe}}{b}\right)T(0)$$

$$g(t) = g(0) - b\dot{x}_d(t) - K_{pe}\left(x_d(t) - x(0)\right) \qquad t > 0$$

Role of the γ -motor neuron drive to the spindle: *g*(*t*) can be programmed based on expected length change in the muscle

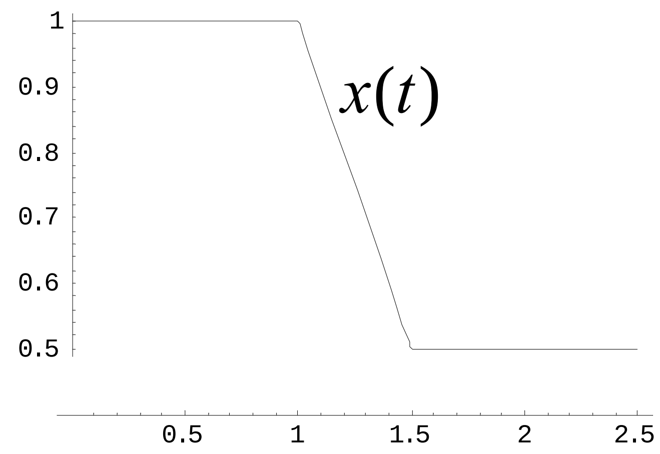
$$\dot{T} = \frac{K_{se}}{b}\left(b\dot{x} + K_{pe}x + g\right) - \left(\frac{K_{se} + K_{pe}}{b}\right)T$$

$$T(0) = \frac{K_{se}(K_{pe}x(0) + g(0))}{K_{se} + K_{pe}}$$

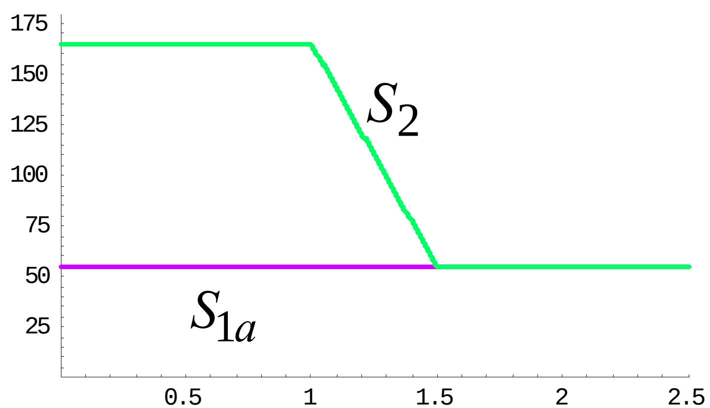
$$g(t) = g(0) - b\dot{x}_d(t) - K_{pe}\left(x_d(t) - x(0)\right)$$

$$S_{1a}(t) = \frac{aT}{K_{se}} \qquad S_2(t) = a\left(x - \frac{T}{K_{se}}\right)$$

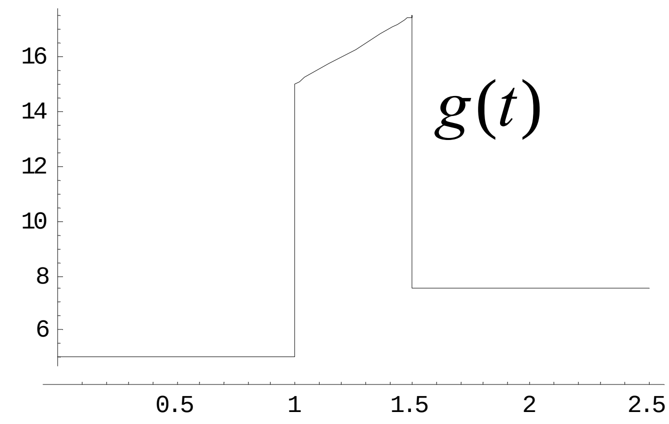
Length change in spindle (cm)



Spindle afferent activity (spikes/sec)



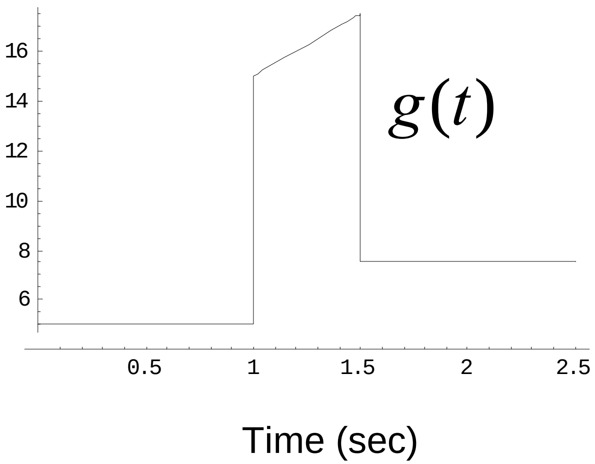
γ -motor neuron input $g(t)$



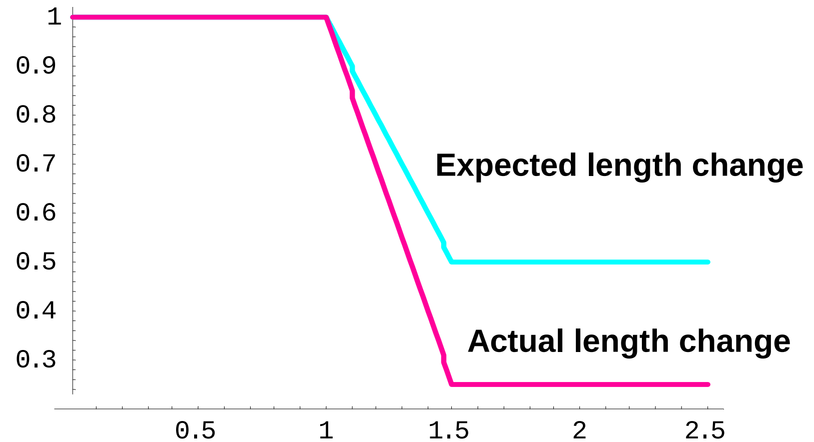
Role of the γ -motor neuron drive to the spindle:
Muscle length change during movement is sensed by afferents
Error can be detected in this feedback

$$g(t) = g(0) - b\dot{x}_d - K_{pe}(x_d - x(0))$$

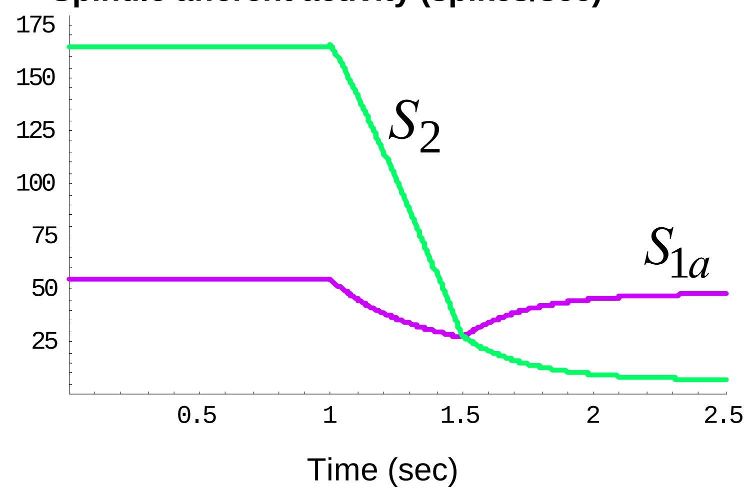
γ -motor neuron input $g(t)$



Length of spindle (cm)



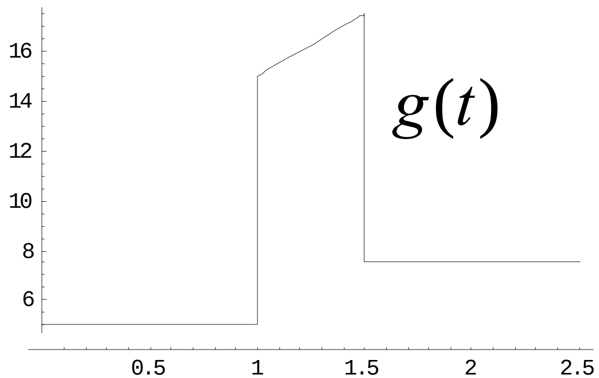
Spindle afferent activity (spikes/sec)



Role of the γ -motor neuron drive to the spindle: spindles can sense error in muscle length change during movement

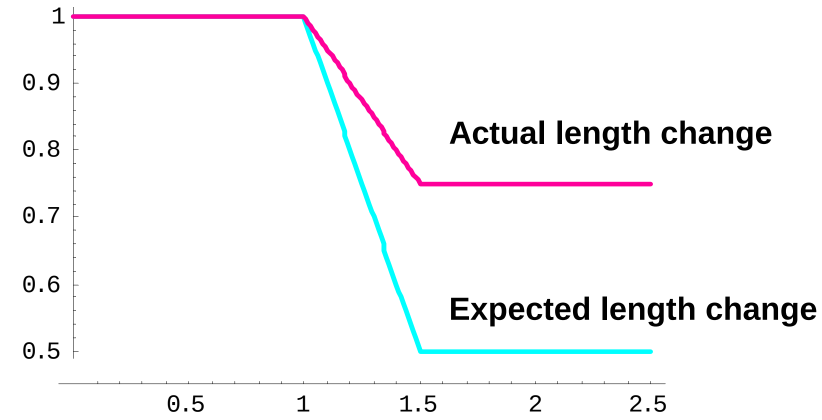
$$g(t) = g(0) - b\dot{x}_d - K_{pe}(x_d - x(0))$$

γ -motor neuron input $g(t)$

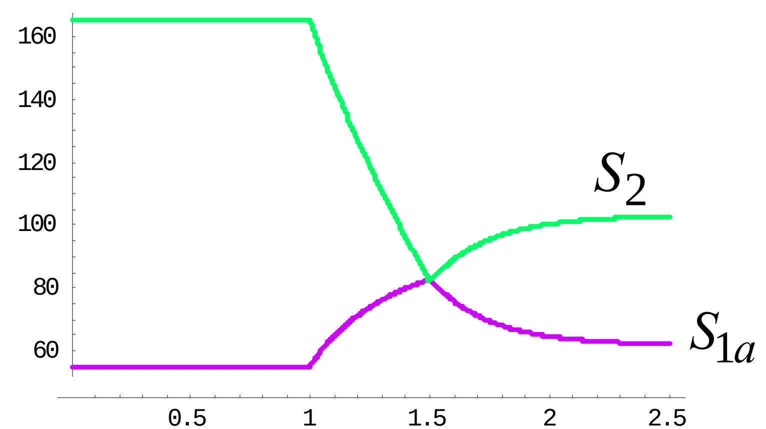


Time (sec)

Length of spindle (cm)

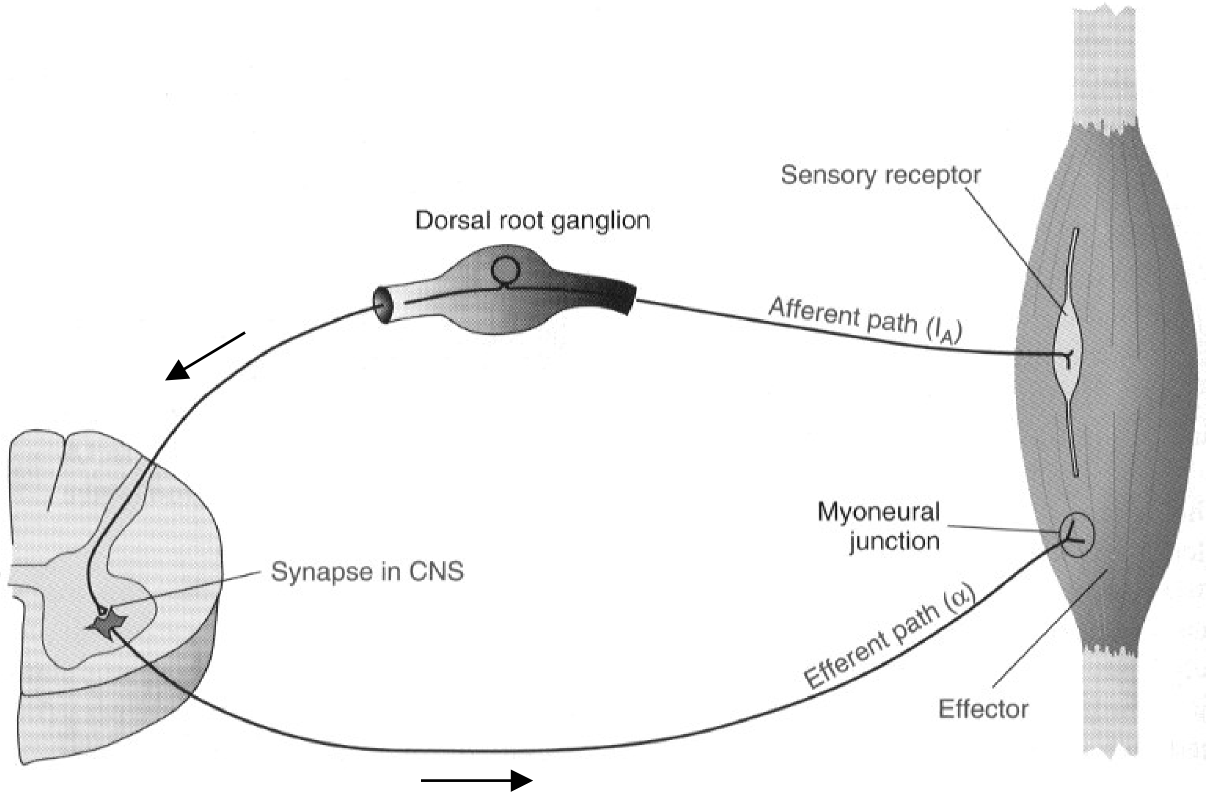


Spindle afferent activity (spikes/sec)



Time (sec)

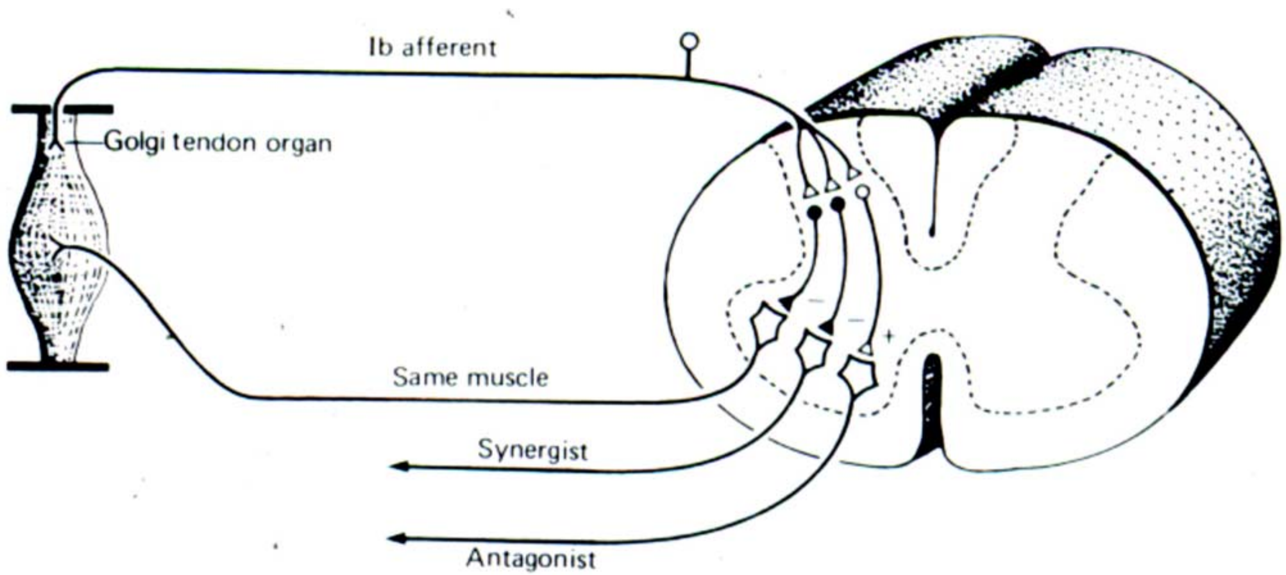
1a spindle afferents excite α-motor neurons of the same muscle mono-synaptically

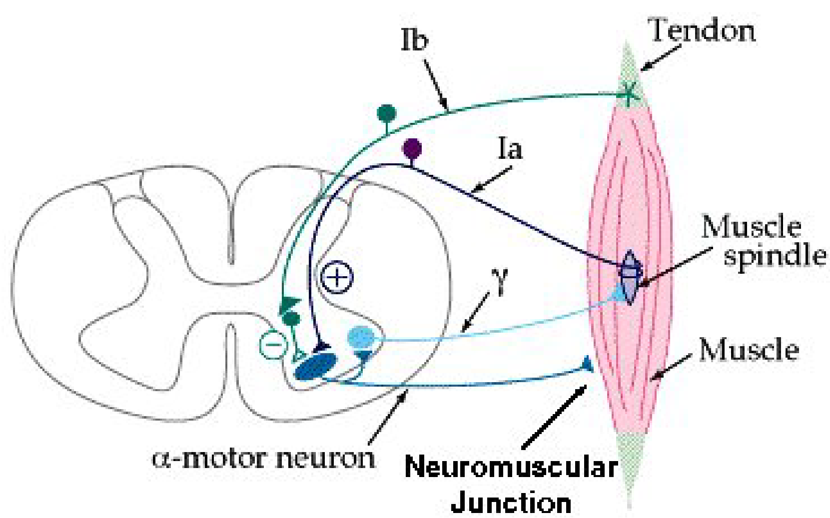


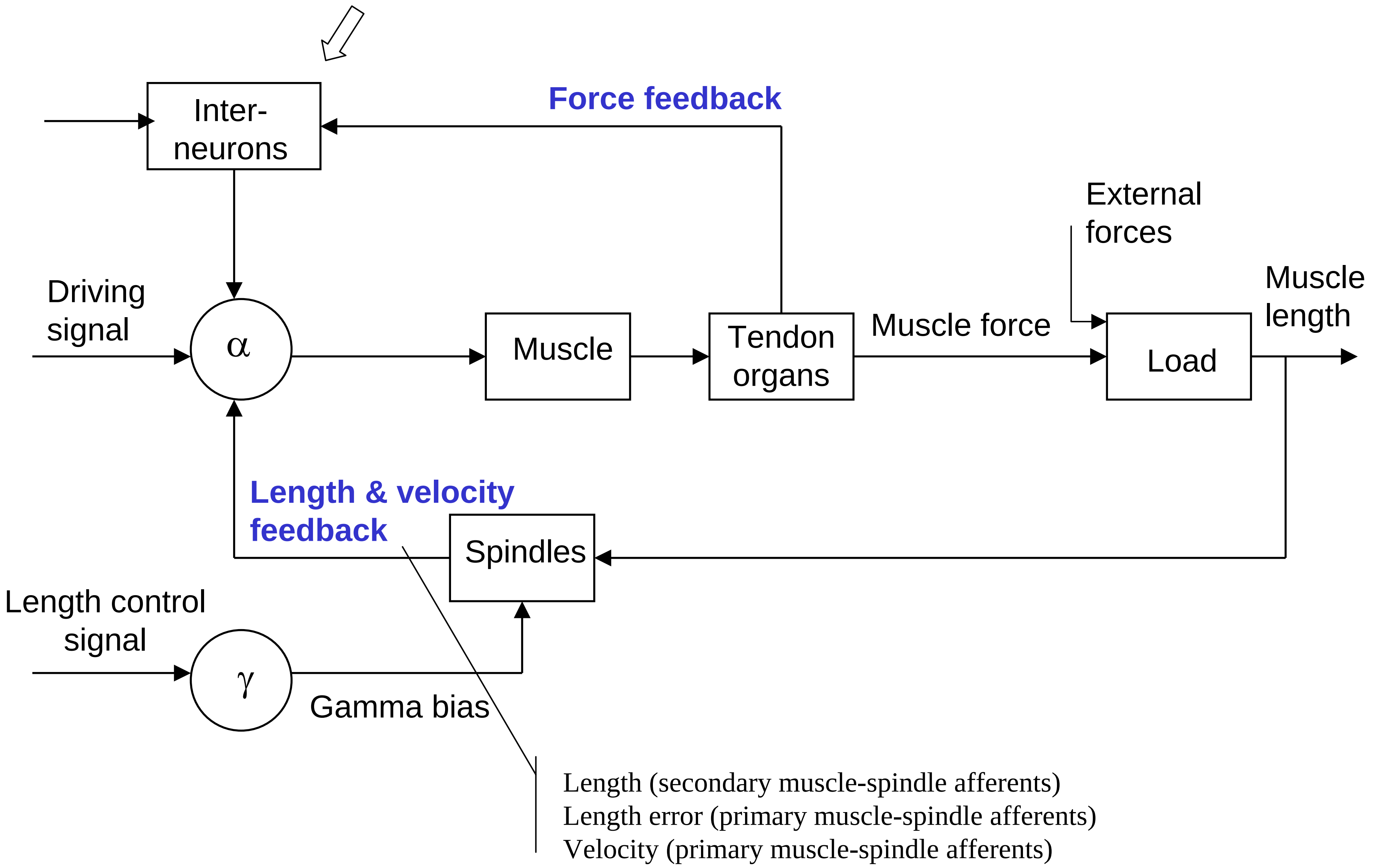
Central connection and function of group II spindle afferents are poorly understood

- Their action on α-motor neurons is via interneurons
- Activity of a group II spindle afferent may excite or inhibit the α- motor neurons (of the same muscle).
- This would suggest that the role of group II spindle afferents depends on the descending commands that are received from the higher centers on the interneurons.

Golgi tendon afferents have an inhibitory effect (via inter-neurons) on α-motor neurons of the same muscle







Inter-
neurons
Force feedback
External
forces
Muscle
length
Driving
signal
α
Muscle
Tendon
organs
Muscle force
Load
Length & velocity
feedback
Spindles
Length control
signal
γ
Gamma bias
Length (secondary muscle-spindle afferents)
Length error (primary muscle-spindle afferents)
Velocity (primary muscle-spindle afferents)