1-1 Forces and torques

1-1A Torque and angular velocity are vector quantities

Imagine that you are holding a ball in one hand on a counter top (Figure.1). If you give your wrist a quick twist, the ball will start rotating. What has happened is that your wrist has imposed a torque on the ball, and the torque has resulted in a motion. This motion is not a translation, meaning a movement from point to point, but is instead rotation, the movement around a pivot. The rotation is described as a change in angle of the ball as a function of time, termed **angular velocity**, which is a vector that points along the axis of rotation of the ball. It has a magnitude that is the speed of rotation, usually expressed in terms of degrees per second or radians per section. Because the direction of this vector can be either up or down, a **right-hand-rule** convention is used to describe the vector’s direction unambiguously. In Figure.1, a curved arrow specifies the direction of rotation of the ball. Imagine the fingers of your right hand curving about this arrow. Your thumb would be pointing down. That is the direction of the angular velocity vector for the rotation of the ball.

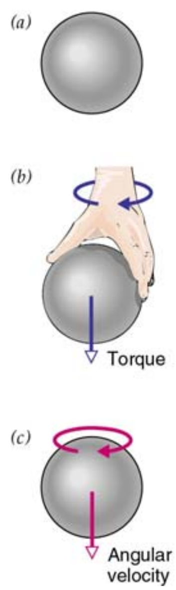

Figure.1. ***Angular velocity and torque.*** *A quick twist of the wrist imposes a torque on the ball that results in the ball acquiring angular velocity. The angular velocity vector is directed along the axis of rotation. Its direction is the same as that of the torque vector. (From L. A. Bloomfield, “How things work: The physics of everyday life”, Wiley, 2001).*

Twisting of the wrist imposed a torque on the ball. This torque is also a vector, and in this simple case its direction is the same as the angular velocity vector, downward.

1-1B Relating muscle forces to joint torques through principle of virtual work

When a muscle such as the triceps contracts, it produces a force that results in a torque on the elbow joint (see Figure.2B). Here we consider how to relate force in a muscle to torque on a joint. In Figure.2A, the force *f* that the muscle produces imposes a torque τ on the joint, resulting in an extension of the joint angle by an amount Δq. We have chosen our angle *q* so that it increases when the joint flexes, which means that in this convention a positive angular velocity results in flexion. Using the right-hand rule, a positive angular velocity is a vector perpendicular to the plane of the page and points away from the reader. Therefore, a positive torque also points away from you and causes flexion of the joint. It is important to note that by selecting the coordinate system to represent angle of this joint, we also specified the coordinate system for torque. Therefore, when we try to represent the muscle's force in terms of torque, the transformation depends on how we define the angle of the joint.

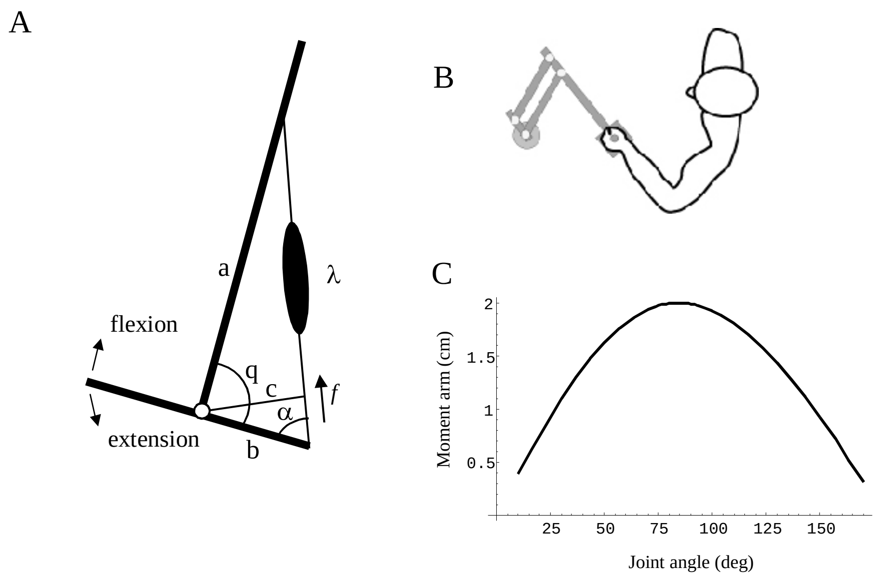

Figure.2. ***Schematic of a muscle that acts on a single joint.*** ***A.*** *Abbreviations: a, b: length of bone from joint to insertion of the triceps muscle; q, the angle supplementary (see* ***Error! Reference source not found.****) to that of the elbow; f, linear force applied by the triceps on the forearm; λ, length of the triceps.* ***B.*** *View from above of a person holding a robotic arm.* ***C.*** *Torque as a function of muscle length. Value of the moment arm Eq. (3) as a function of joint angle for Figure.2A, where a=20 cm, b=2 cm. The value of the moment arm is related to joint torque, whereas the value of joint angle is a function of muscle length*

If we image the muscle acting along a line and label the length of the muscle as λ, the contraction may cause the muscle to shorten by amount $\Delta\lambda$. When the muscle shortens, the **work** that it performs is the force (*f*) that it produces times the displacement that it undergoes. The muscle shortens when it contracts. Therefore, $\Delta\lambda$ has a negative value. When the muscle changes length by amount $\Delta\lambda$, the joint rotates by amount Δq. The work that the muscle performs in shortening its length is the same as the **virtual work** it performs in rotating the joint. To signify the fact that the muscle performed positive work while it shortened, we define the work that it performed as:

$$\text{work in the muscle} = -f\,\Delta\lambda$$

The work that it performs in rotating the joint is:

$$\text{work in the joint} = \tau\,\Delta q$$

And because these quantities are equal, we have:

$$\tau\,\Delta q = -f\,\Delta\lambda$$

$$\tau = -\frac{\Delta\lambda}{\Delta q} f$$

If we allow the length and angular changes to approach zero, i.e., we make each change step miniscule, we can represent them as a derivative:

$$\tau = -\frac{d\lambda}{dq} f \tag{1}$$

The derivative in the above relation describes how the length of the muscle changes with respect to a change in the joint angle. It expresses an important idea: The torque that a muscle produces on a joint depends on how its length changes with respect to the angle of the joint. This notion also explains the concept of virtual work: it is virtual because the amount is infinitesimal as the steps approach zero. Because of the choice of reference frame of in this example, an increase in the angle q results in an increase in the length of the muscle. (Note that the angle in question is complementary to that formed by the upper arm and the lower arm. Accordingly, we could as easily describe the movement with a reference frame in which a decrease in angle results in an increase in muscle length.) To express the derivative in Eq. (1), we need to consider the geometry of the limb. Consider the triangle that is composed of the muscle length λ and lengths a and b in Figure.2. We can express how the length of the muscle depends on the angle of the joint through the law of cosines:

$$\lambda = \sqrt{a^2 + b^2 - 2ab\cos(q)} \tag{2}$$

The derivative of this function is:

$$\frac{d\lambda}{dq} = \frac{ab\sin(q)}{\sqrt{a^2 + b^2 - 2ab\cos(q)}} \tag{3}$$

Inserting Eq. (3) in Eq. (1) specifies how force in the muscle is related to torque on the joint. The larger the value of the function in Eq. (3), the larger the magnitude of torque that a given force in the muscle will produce. Figure.2C plots how Eq. (3) varies with respect to joint angle q for given values of a and b. In this example, the muscle exerts its greatest torque on the joint when the angle is approximately 84 degrees. At very flexed or very extended positions, the function becomes small. This means that at these joint angles the muscle force results in very little joint torque.

1-1C Moment arms

In the previous section, we used the principle of virtual work to derive the relationship in Eq. (1). Now let us show that Eq. (2) is in fact the familiar moment arm that relates forces to torques. The moment arm of the muscle in Figure.2 is length *c*. It is the length *c* that connects the center of rotation of the joint with a line perpendicular to the line of action of the muscle. The length of the moment arm is:

$$c = b\sin(\alpha)$$

From the law of sines, we know that:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(q)}{\lambda}$$

which, using Eq. (2), gives us an expression for the length of the moment arm:

$$c = \frac{ab\sin(q)}{\lambda} = \frac{ab\sin(q)}{\sqrt{a^2 + b^2 - 2ab\cos(q)}}$$

This is identical to the expression that we derived in Eq. (3) from the principle of virtual work. That is, the change in muscle length λ with respect to joint angle q is the moment arm of this single-joint, single-muscle system:

$$\frac{d\lambda}{dq} = c$$

Both methods produced a function that we can use to relate muscle force to torque. However, the principle of virtual work is a powerful method that is worth exploiting because it can aid us when muscles have complex geometry. We take up this issue in the next section.

1-1D Multiple-joint muscles

Contraction of a muscle can result in torques on multiple joints. For example, consider the muscle in Figure 3. How does force in this muscle relate to torques on the joints? We can use the principle of virtual work to describe this relationship.

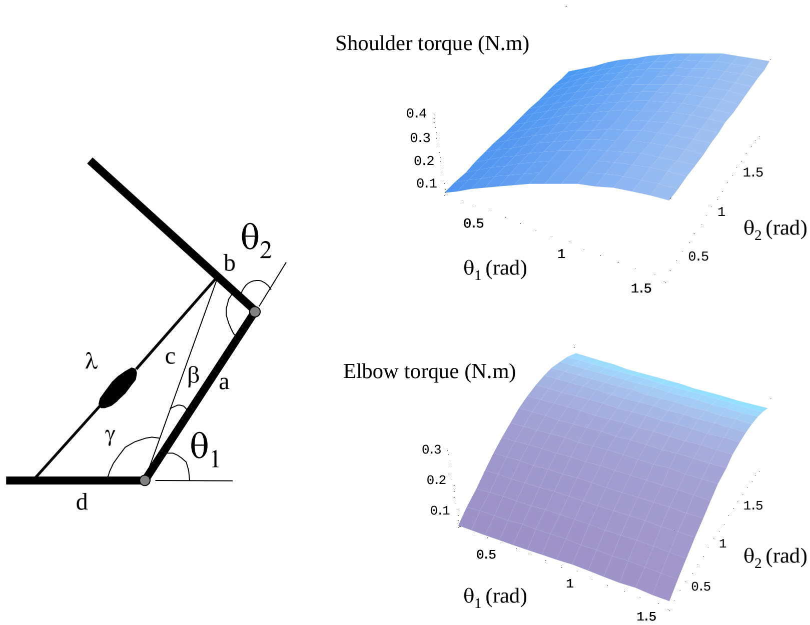

Figure 3. ***A two-joint muscle****. The torque on the shoulder (θ_1) and elbow (θ_2) is plotted as a function of angle of each joint for a constant muscle force of 10 Newtons (N). Link lengths: a=0.33, d=0.04, b=0.03m.*

Work is a scalar quantity that can be expressed as the dot product of a force vector and a difference vector, the analogue to work equals force times distance. We can also represent that same work as a torque vector that produces a change in the angles of a system. Now in the case that we are considering here, our muscle produces a force that causes it to shorten in length, which also produces a torque on a joint and rotates it. The work that is done in shortening is the same as the work that is done in rotating the limb. The work in the muscle is represented as the dot product of its force and length change. The work in the joint is represented as the dot produce of torque and the joint’s angular change. However, because the muscle shortens when it produces its force, the dot product of force and length change is negative. To take care of this, we write:

$$\tau^T \Delta q = -f^T \Delta\lambda \tag{4}$$

In this equation, the force and torque may be multi-dimensional vectors. If we allow the displacements to become infinitesimal, we can define a **Jacobian matrix** *J*, termed simply a Jacobian. The Jacobian here is the derivative of the length change with respect to the change in joint angle:

$$J = \frac{d\lambda}{dq} \tag{5}$$

Using the Jacobian, we have:

$$\Delta\lambda = J\Delta q$$

Inserting this into Eq. (4) gives:

$$\tau^T \Delta q = -f^T J \Delta q,$$

which must hold true for all Δq, and therefore we have:

$$\tau^T = -f^T J$$

Transposing both sides yields the result:

$$\tau = -J^T f \tag{6}$$

This is an interesting relationship because it allows us to convert a force in muscle coordinates into a torque in joint coordinates. The relationship depends on how the length of the muscle changes with respect to the joint angle. If the length of the muscle depends on multiple joint angles, then the Jacobian will be a multi-dimensional vector.

To better understand this relationship, consider again the muscle illustrated in Figure 3. Here, the length of the muscle λ depends on angles θ_1 and θ_2. To find the Jacobian, we need to first express this dependence and then find its derivative. We begin by writing the length c in terms of the angle θ_2:

$$c = \sqrt{a^2 + b^2 + 2ab\cos(\theta_2)} \tag{7}$$

Next we use the law of sines to express angle β and then write length λ in terms of it:

$$\beta = \arcsin\left(\frac{b\sin(\theta_2)}{c}\right)$$

$$\lambda = \sqrt{d^2 + c^2 + 2dc\cos(\beta + \theta_1)}$$

Using the identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we get:

$$\lambda = \sqrt{d^2 + c^2 + 2dc[\cos(\beta)\cos(\theta_1) - \sin(\beta)\sin(\theta_1)]} \tag{8}$$

Next, we note that in Eq. (8), we have a $\cos(\beta)$ term and β contains an arcsin term. We know that $\cos^2 x = 1 - \sin^2 x$. If we set $x = \arcsin(y)$, we get the expression $\cos(\arcsin(y)) = \sqrt{1-y^2}$. We can use this identity to replace the expression for β in Eq. (8) with the following:

$$\lambda = \sqrt{d^2 + c^2 + 2dc\left(\sqrt{1 - \frac{b^2\sin^2(\theta_2)}{c^2}}\cos(\theta)_1 - \frac{b\sin(\theta_2)}{c}\sin(\theta_1)\right)} \tag{9}$$

After we insert the expression for c from Eq. (7) into the above relation, we are left with an expression for λ that depends on lengths a, b, d, and angles θ_1 and θ_2. To find the relationship between force in the muscle and the torques on each joint, we find the Jacobian, which in this case is a 1 x 2 vector:

$$J = \frac{d\lambda}{d\boldsymbol{\theta}} = \begin{bmatrix} \frac{d\lambda}{d\theta_1} & \frac{d\lambda}{d\theta_2} \end{bmatrix}$$

If we insert this vector into Eq. (6), we can compute the torques on each joint for a given muscle force. The 1 x 1 force vector produces a 2 x 1 torque vector. In Figure 3 we have plotted how a 10-N force in the muscle is distributed to each joint. We see that the maximum torque is produced when both joints are in their flexed posture (large values for θ_1 and θ_2). In the extended posture, the muscle cannot produce a significant torque on either joint.

Note, however, that while the torque on the shoulder (θ_1) increases as the shoulder joint flexes (θ_1 increases), the torque on the shoulder does not change very much as the elbow joint rotates. In comparison, torque on the elbow strongly depends on θ_2, but does not change very much as θ_1 changes. Therefore, although this muscle acts on two joints, its moment arm on one joint does not depend strongly on the position of the other joint.

1-1E Torques depend on the coordinates in which joint angles are represented

Eq. (6) expresses the idea that when a muscle contracts, the work that it performs is the same whether we measure that work in terms of a force acting through a displacement in muscle length or a torque acting through a displacement in angular coordinates. The quantity that allows us to relate forces to torques is the Jacobian. The Jacobian describes how length changes in one coordinate system are dependent on changes in another coordinate system. If we change the coordinate system in which we represent the angles of the limb, we should expect that the Jacobian would change. Because of this, the way in which torques are described will also change. This leads to the counterintuitive conclusion that the force that our muscle produces will generate different torques depending on how we choose to measure the angle of that joint. Is this true? It is.

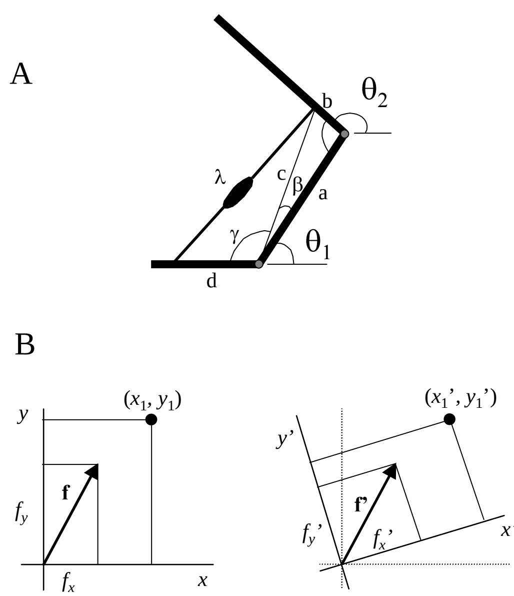

Figure 4. ***A. A two joint muscle acting on a two-link system****. The system is identical to Figure 3 except that the angle of the second link is measured with respect to an absolute axis, rather than relative to the angle of the first link.* ***B. Rotation of a coordinate system in which the position of a point is described***

***changes the representation of force as well**. Left: the position of a point and a force acting on it is represented in coordinate system x. We rotate this coordinate system to get x’ (Right). Representation of force changes from f to f’.*

An example will illustrate this point: Consider the muscle in the two-link system in Figure 4A. Here, we choose to measure the angle of the second link θ_2 in a slightly different way than we did in Figure 3. Whereas in the earlier example the angle of the second link was measured with respect to the orientation of the first link, i.e., in a “relative” coordinate system, here the angle is measured with respect to an “absolute” coordinate. Therefore, θ_2 in Figure 3 is equal to $\theta_2 - \theta_1$ in Figure 4. Otherwise, everything is identical in these two systems. To express how the length of the muscle in Figure 4A depends on the angles of the system, we replace θ_2 in Eq. (7) and (9) with $\theta_2 - \theta_1$:

$$c = \sqrt{a^2 + b^2 + 2ab\cos(\theta_2 - \theta_1)}$$

$$\lambda = \sqrt{d^2 + c^2 + 2dc\left(\sqrt{1 - \frac{b^2 \sin^2(\theta_2 - \theta_1)}{c^2}}\cos(\theta_1) - \frac{b\sin(\theta_2 - \theta_1)}{c}\sin(\theta_1)\right)}$$

If we now find the Jacobian $[d\lambda / d\theta_1 \quad d\lambda / d\theta_2]$, we can transform forces in the muscle to torques in this new coordinate system. For example, consider a condition where the muscle is producing a 10-N force and the links in Figure 4A are positioned at $\theta_1=45^o$ and $\theta_2=90^o$. When the Jacobian is evaluated at these joint angles, it produces the vector $[-0.0061 \quad -0.0216]$ with units of meters/radian. Therefore, the 10-N muscle force produces a torque of 0.061 N-m on joint 1 and a torque of 0.216 N-m on joint 2. Let us now compare this to the joint torques that are produced when the same force acts on the links in Figure 3. For the links to be at the same physical location in Figure 3 and Figure 4A, angles for Figure 3 are: $\theta_1=45^o$ and $\theta_2=45^o$. The Jacobian for the system of Figure 3 at these angles is the vector $[-0.0277 \quad -0.0216]$. The 10-N force in the muscle of the system illustrated in Figure 3 produces a torque of 0.277 N-m on joint 1 and a torque of 0.216 N-m on joint 2. The same 10 N of force in the muscle of Figure 4A produced an identical torque on joint 2, but a different torque on joint 1. Therefore, we see that when we changed the representation of position for the second joint from Figure 3 to Figure 4A, the change did not affect the measure of torque on that joint, but it affected the torque on the first joint.

Although the change in representation of position has changed the torques on the joints, intuition tells us that the limbs in Figure 3 and Figure 4A should move identically if the muscle produces the same force. Of course, their positions in the two different coordinate systems will be different, but if we were looking at the limb when the force is applied, we would see it move the same regardless of whether we choose to represent position using one or the other method. Why is this?

To help understand what happened, it is worth thinking a bit about what forces and torque are. In Newton’s second law, force is related linearly to acceleration $\mathbf{f} = m\ddot{\mathbf{x}}$. Here, **f** is a 3 dimensional vector with components along the x-, y-, and z-axis, and so this equation is really 3 different equations. The coordinate system in which position of the object is described (**x**) is the same coordinate system in which forces (**f**) are described. If we change the coordinate system for position, the force vector will also change.

Consider Figure 4B, where position of a point is described in terms of a coordinate system (*x*, *y*). Note that when we make a change to this coordinate system (by rotating it in this example from *x*, *y* to *x*’, *y*’), the representation of both the position of the point and the force changes.

We usually use a cartesian coordinate system to describe forces. However, when we wanted to represent force in a muscle, we simplified things by assuming that the muscle was basically a line and that force in the muscle acted along that line. So in this case, force was a one-dimensional quantity because “position of our muscle” was simply its length, which is also a one-dimensional quantity.

When we represented position of our limb in terms of the angles of the joints, we wished to know how we could represent the forces in the muscle in terms of forces in this new coordinate system of joint angular positions. We used the principle of virtual work to relate forces to torques: a Jacobian that describes how positions in muscle coordinates relate to positions in joint coordinates also describes how forces in muscle coordinates relate to forces in joint coordinates. We gave the name torque to the representation of force in joint coordinates. When we changed the way we measured position in joint coordinates, we implicitly were also changing the way that the torque vector was represented in that coordinate system.

Now the remarkable thing is that it does not really matter how we choose to measure the position of an object. In fact, any choice that we have made is simply for the sake of convenience. The motion that the object will follow for a given force (that force properly defined in the coordinate system in which we chose to represent position) is the same to an independent, external observer. Therefore, from the point of view of Newton’s 2nd law, there is no advantage in choosing one coordinate system to another. However, when we choose to represent position of the limb by picking a particular coordinate system, we have implicitly also chosen the coordinate system in which forces will be represented.